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Characterization for Cauchy Continuity and Sufficiency of Sequences

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ABSTRACT

A function from a metric space to a metric space preserves Cauchy sequences if and only if it is uniformly continuous on every totally bounded subset of the domain; or equivalently, if and only if it is uniformly continuous on every subset whose members are the terms of a Cauchy sequence in the domain. This characterization is established in this article. Possible compromises to replace nets by sequences are also discussed. A linear mapping from a metrizable topological vector space to a metrizable topological vector space is continuous if and only if it preserves Cauchy sequences. This fact is also observed as an application of the first characterization.

Keywords**:** Cauchy sequence, totally bounded set, uniform continuity, topological algebra.

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1. INTRODUCTION

A mapping from a metric space to a metric space is continuous if and only if image of any convergent sequence is a convergent sequence and the image of the limit of the convergent sequence in the domain is the limit of the convergent sequence in the co domain. This statement may be considered either as a definition or as a characterization of a continuous function. The definition of a Cauchy continuous function [3, 6, 20] is the following. A mapping from a metric space to a metric space is said to be Cauchy continuous if and only if image of any Cauchy sequence is a Cauchy sequence. There are many research articles and books on Cauchy continuous functions. But, it seems that a characterization for Cauchy continuous functions is missing. So, every characterization of Cauchy continuous functions is important. This present article establishes that a mapping from a metric space to a metric space is Cauchy continuous if and only if it is uniformly continuous on every totally bounded subset of the domain. Both concepts of uniform continuity and totally bounded subsets require uniformity on uniform spaces. Metric spaces are uniform spaces, and the uniformities are induced by the metric on a metric space. But, all uniform spaces need not be metric spaces in the sense that all uniformities cannot be derived from single metrics. So, the problems of finding a definition for Cauchy continuity and finding a characterization for Cauchy continuity for functions from a uniform space to a uniform space are being difficult ones. There is an article [8] which discusses functions which preserve Cauchy nets from uniform spaces to uniform spaces. A function from a metric space to a metric space preserving Cauchy nets need not preserve Cauchy sequences, and a function from a metric space to a metric space preserving Cauchy sequences need not preserve Cauchy nets. So, there is a problem for finding a generalized definition for Cauchy continuity of functions on uniform spaces. There is an article [21] which discusses functions which preserve Cauchy filters from uniform spaces to uniform spaces. There are articles [9, 11, 17, 19, 23] which discuss functions which preserve Cauchy series from topological vectors to topological vector spaces, where topological vector spaces are uniform spaces with algebraic operations. The present article uses the characterization of Cauchy continuity to extend a continuous Cauchy continuous function from a metric space to a metric space continuously from the completion of the domain to the completion of the co domain. The present article observes that the points in closures of subsets of uniform spaces can be described by means of sequences instead of nets. The present article also proposes a method to avoid nets and to use sequence to describe some nature of functions from uniform spaces to uniform spaces. The article ends with finding sufficient conditions for mapping bounded sets to bounded sets by using Cauchy continuity.

2. PSEUDO METRICS

Let us begin with the definition of pseudo metrics [5] to fix terminologies. Let X be a non empty set. Let $d: X \times X \to [0, \infty)$ be a non negative function. It is called a pseudo metric on X, if $d(x,x) = 0$, $d(x,y) = d(y,x)$, and $d(x,y) \leq d(x,z) + d(z,y)$, $\forall x, y, z \in X$. This pseudo metric is called a metric on X, if $x = y$, whenever $d(x,y) = 0$. The collection of all subsets of $X \times X$ containing sets of the form $\{(x,y) \in X \times X: d(x,y) < r\}$, $r > 0$, is called the (natural) uniformity on X induced by a pseudo metric d on X. The collection of all subsets of X which are the unions of sets of the form $\{y \in X: d(x,y) < r\}$, $r > 0$, $x \in X$, along with the empty subset, is called the (natural) topology on X induced by a pseudo metric d on X. The natural topology and the natural uniformity induced by a metric are Hausdorff. For definitions of concepts of topology and uniformity and for related concepts, let us refer to the book General Topology of J.L. Kelly.

Let (X, d) be a pseudo metric space. A sequence $(x_n)_{n=1}^{\infty}$ is said to be converging to x_0 in (X, d), when $d(x_n, x_0) \to 0$ as $n \to \infty$. A sequence $(x_n)_{n=1}^{\infty}$ is said to be a Cauchy sequence [7] in (X, d), when $d(x_n, x_m) \to 0$ as $n, m \to \infty$. A pseudo metric space is called a complete pseudo metric space, if every Cauchy sequence in the pseudo metric space converges in it. A subset A of a pseudo metric space (X, d) is said to be totally bounded [10], if for any given $\varepsilon > 0$, there are finitely many elements $x_1, x_2, \dots, x_n$ in X such that for every $y \in A$, there is some x_i such that $d(x_i, y) < \varepsilon$. If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X, d), then the set $\{x_n : n = 1, 2, 3, \dots\}$ is a totally bounded subset of (X, d).

Let $f: (X, d_X) \to (Y, d_Y)$ be a function from a pseudo metric space into a pseudo metric space. Let $x_0 \in X$. The function f is said to be continuous at x_0, if for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, x_0) > 0$ such that $d_Y\big(f(x), f(x_0)\big) < \varepsilon$ whenever $d_X(x, x_0) < \delta$ with $x \in X$. The function f is said to continuous on X, if f is continuous at every point of X. The function f is said to be uniformly continuous on X, if for every $\varepsilon > 0$, there is a common $\delta = \delta(\varepsilon) > 0$ such that $d_Y\big(f(x), f(x_0)\big) < \varepsilon$ whenever $d_X(x, x_0) < \delta$ with $x, x_0 \in X$. The function f is said to be uniformly continuous [1, 2, 4] on a subset A of X, if for every $\varepsilon > 0$, there is a common $\delta = \delta(\varepsilon) > 0$ such that $d_Y\big(f(x), f(x_0)\big) < \varepsilon$ whenever $d_X(x, x_0) < \delta$ with $x, x_0 \in A$. If f is uniformly continuous on a totally bounded subset A of (X, d_X), then $f(A)$ is a totally bounded subset of (Y, d_Y). If f is uniformly continuous on (X, d_X), then $\big(f(x_n)\big)_{n=1}^{\infty}$ is a Cauchy sequence in (Y, d_Y) whenever $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X, d_X).

Let (X, d) be a metric space. Two Cauchy sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are said to be equivalent, written as $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$, if $d(x_n, y_n) \to 0$ as $n \to \infty$, or equivalently, if $d(x_n, y_m) \to 0$ as $n, m \to \infty$. This is an equivalence relation in the collection of all Cauchy sequences of (X, d). The equivalence class containing a Cauchy sequence $(x_n)_{n=1}^{\infty}$ is denoted by $[(x_n)_{n=1}^{\infty}]$. Let X_c denote the collection of all such equivalence classes. For any two such equivalence classes $[(x_n)_{n=1}^{\infty}]$ and $[(y_n)_{n=1}^{\infty}]$, let us write $d([(x_n)_{n=1}^{\infty}], [(y_n)_{n=1}^{\infty}])$ for the limit value $\lim_{n\to\infty} d(x_n, y_n)$. Then (X_c, d) is called the completion of (X, d). Here (X_c, d) is a complete metric space. Let us consider the mapping $I: (X, d) \to (X_c, d)$ defined by $I(x) = [(x_n)_{n=1}^{\infty}]$, where $x_n = x, \forall n$, for every $x \in X$. Then I is an isometric mapping, and $I(X)$ is dense in (X_c, d). Moreover, for any given Cauchy sequence $(x_n)_{n=1}^{\infty}$ in (X, d), the sequence $\big(I(x_n)\big)_{n=1}^{\infty}$ converges to $[(x_n)_{n=1}^{\infty}]$ in (X_c, d). Thus, X may be considered as a dense subspace of a complete metric space (X_c, d). There may be many complete pseudo metric spaces (Y, d) in which a given pseudo metric space (X, d) becomes dense subspaces. However, if (X, d) is a metric space, then there is a unique complete metric space (Y, d) in which (X, d) becomes a dense subspace; uniqueness up to an isometric mapping.

Let (X, d_X) and (Y, d_Y) be two metric spaces with the completions (X_c, d_X) and (Y_c, d_Y), respectively. If $f: (X, d_X) \to (Y, d_Y)$ is a uniformly continuous mapping, then there is a natural unique continuous extension $f_c: (X_c, d_X) \to (Y_c, d_Y)$.

3. NEW CHARACTERIZATION

Let us recall that a mapping $f:(X,d_X)\to(Y,d_Y)$ is Cauchy continuous, if $\left(f(x_n)\right)_{n=1}^{\infty}$ is Cauchy in the pseudo metric space (Y,d_Y), whenever $(x_n)_{n=1}^{\infty}$ is Cauchy in the pseudo metric space (X,d_X).

Theorem 1: Let $T:(X,d_X)\to(Y,d_Y)$ be a mapping between pseudo metric spaces. Then the followings are equivalent. (1) $T:(X,d_X)\to(Y,d_Y)$ is Cauchy continuous. (2) T is uniformly continuous on every totally bounded subset of (X,d_X). (3) T restricted to the set $\{x_n: n=1,2,3,\ldots\}$ is uniformly continuous for every Cauchy sequence $(x_n)_{n=1}^{\infty}$ in (X,d_X).

Proof: Suppose that (2) is not true. Let B be a totally bounded subset of (X,d_X) such that T restricted to B is not uniformly continuous. Then there is an $\varepsilon>0$ such that there is no $\delta>0$ such that $d_Y(Tx,Tx")<\varepsilon$ whenever $d_X(x,x")<\delta$ in B, that is when $x,x"\in B$. Since B is totally bounded, it is covered by finitely many open balls with centres in B and with radius $\frac{1}{2^1}$. Then there is one ball $B_1=\left\{x\in B: d_X(x_1,x)<\frac{1}{2^1}\right\}$ with centre x_1 in B such that there is no $\delta>0$ such that $d_Y(Tx,Tx")<\varepsilon$ whenever $d_X(x,x")<\delta$ in B_1. In general, there is one ball $B_n=\left\{x\in B_{n-1}: d_X(x_n,x)<\frac{1}{2^n}\right\}$ with centre x_n in B_{n-1} such that there is no $\delta>0$ such that $d_Y(Tx,Tx")<\varepsilon$ whenever $d_X(x,x")<\delta$ in B_n, for $n=2,3,4,\ldots$. There are x_n' and x_n'' in the ball B_n such that $d_Y(Tx_n',Tx_n'')\geq\varepsilon$, for every $n=1,2,3,\ldots$. Here, $d_X(x_n',x_n'')<\frac{1}{2^{n-1}}$, $d_X(x_{n+1}',x_n'')<\frac{1}{2^{n-1}}$, $d_X(x_n',x_{n+1}'')<\frac{1}{2^{n-1}}$, for every $n=1,2,3,\ldots$. Because of these inequalities, it follows that the sequence $x_1',x_1'',x_2',x_2'',x_3',x_3'',\ldots$ is a Cauchy sequence in (X,d_X). But, $Tx_1',Tx_1'',Tx_2',Tx_2'',Tx_3',Tx_3'',\ldots$ is not a Cauchy sequence in (Y,d_Y), because $d_Y(Tx_n',Tx_n'')\geq\varepsilon$, for every $n=1,2,3,\ldots$. So, (1) $\Rightarrow$ (2) is true. Since $\{x_n: n=1,2,3,\ldots\}$ is totally bounded whenever $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X,d_X), then (2) $\Longrightarrow$ (3) is also true. Assume that (3) is true. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in (X,d_X). Then T restricted to the totally bounded subset $\{x_n: n=1,2,3,\ldots\}$ is uniformly continuous. Fix $\varepsilon>0$. Then there is a $\delta>0$ such that $d_X(x_n,x_m)<\delta$ implies $d_Y(Tx_n,Tx_m)<\varepsilon$. Find an integer n_0 such that $d_X(x_n,x_m)<\delta, \forall n,m\geq n_0$. Then $d_Y(Tx_n,Tx_m)<\varepsilon, \forall n,m\geq n_0$. This proves that $(Tx_n)_{n=1}^{\infty}$ is a Cauchy sequence in (Y,d_Y). So, (3) $\Rightarrow$ (1) is true. This completes the proof of the theorem.

This theorem is useful in deriving a continuous extension of a continuous Cauchy continuous function between two metric spaces to the completions of these metric spaces. It is known that such extensions are possible for uniformly continuous functions.

4. CONTINUOUS EXTENSION

Let us use the notations mentioned in Section 2 for the completions of metric spaces.

Theorem 2: Let (X,d_X) and (Y,d_Y) be two metric spaces with the respective completions (X_c,d_X) and (Y_c,d_Y). Let $T:(X,d_X)\to(Y,d_Y)$ be a given continuous mapping which is also a Cauchy continuous mapping. Then there is a unique continuous function $T_c:(X_c,d_X)\to(Y_c,d_Y)$ such that it is an extension of the mapping $T:(X,d_X)\to(Y,d_Y)$.

Proof: Let us consider two equivalent Cauchy sequences $(x_n)_{n=1}^{\infty}$ and $(x'_n)_{n=1}^{\infty}$ in (X, d_X). Then $d_X(x_n, x'_n) \to 0$ as $n \to \infty$, and $(Tx_n)_{n=1}^{\infty}$ and $(Tx'_n)_{n=1}^{\infty}$ are Cauchy sequences in (Y, d_Y). Let $A = \{x_n, x'_n : n = 1, 2, 3, \ldots\}$. This is a totally bounded subset of (X, d_X) so that the restriction of T to A is uniformly continuous. Let $\varepsilon > 0$ be given. Find a $\delta > 0$ such that $d_Y(Tx, Tx') < \varepsilon$ whenever $d_X(x, x') < \delta$ with $x, x' \in A$. Find an integer n_0 such that $d_X(x_n, x'_n) < \delta$, $\forall n \geq n_0$. Then $d_Y(Tx_n, Tx'_n) < \varepsilon$, $\forall n \geq n_0$. So, $(Tx_n)_{n=1}^{\infty}$ and $(Tx'_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences in (Y, d_Y). Therefore the mapping $T_c: (X_c, d_X) \to (Y_c, d_Y)$ defined by $T_c([(x_n)_{n=1}^{\infty}])$=$[(Tx_n)_{n=1}^{\infty}]$, $\forall [(x_n)_{n=1}^{\infty}] \in X_c$ is a well defined function and that this function is an extension of the continuous mapping $T: (X, d_X) \to (Y, d_Y)$. The continuity of T_c is yet to be established.

Let $(x_n)_{n=1}^{\infty}$ be a sequence in (X_c, d_X) that converges to some x_0 in (X_c, d_X). To each n, let $\left(x_{n_m}\right)_{m=1}^{\infty}$ be a Cauchy sequence in (X, d_X) such that it converges to x_n in (X_c, d_X). By the previous part of this proof, it follows that $x_{n_m} \to x_n$ and $Tx_{n_m} \to T_c x_n$ as $m \to \infty$. For each n, let us find a member x_{n_m}, say x'_n such that $d_X(x_n, x'_n) < \frac{1}{n}$, and such that $d_Y(T_c x_n, Tx'_n) < \frac{1}{n}$. Let $\varepsilon > 0$ be given. Let k_0 be an integer such that $d_X(x_n, x_0) < \varepsilon$, $\forall n \geq k_0$. Then $d_X(x'_n, x_0) \leq d_X(x'_n, x_n) + d_X(x_n, x_0) < \frac{1}{n} + \varepsilon$, $\forall n \geq k_0$. Then $Tx'_n \to T_c x_0$ as $n \to \infty$, by the previous part of this proof. Find an integer k_1 such that $d_Y(Tx'_n, T_c x_0) < \varepsilon$, $\forall n \geq k_1$. Then, $d_Y(T_c x_n, T_c x_0) \leq d_Y(T_c x_n, Tx'_n) + d_Y(Tx'_n, T_c x_0) < \frac{1}{n} + \varepsilon$, $n \geq k_1$. This proves that the sequence $(T_c x_n)_{n=1}^{\infty}$ converges to $T_c x_0$ in (Y_c, d_Y). This proves the continuity of T_c. Uniqueness of the extension follows from the construction of T_c. This completes the proof.

5. SOME EXAMPLES

Theorem 2 poses many natural questions. Some possible partial answers are the followings. The first one is a corollary to Theorem 1.

Corollary 1: Let $T: (X, d_X) \to (Y, d_Y)$ be a Cauchy continuous mapping between two pseudo metric spaces. Suppose that a point x_0 of (X, d_X) has a totally bounded neighbourhood. Then T is continuous at x_0.

Corollary 2: Let $T: (X, d_X) \to (Y, d_Y)$ be a Cauchy continuous mapping between two pseudo metric spaces. Suppose that every point of (X, d_X) has a totally bounded neighbourhood. Then T is continuous on (X, d_X).

Example 1: Every point of every finite dimensional Euclidean space with the usual Euclidean metric has a totally bounded neighbourhood.

Definition 1: Let $T: (X, d_X) \to (Y, d_Y)$ be a mapping between two pseudo metric spaces. Consider a point x_0 of (X, d_X). If $(Tx_n)_{n=1}^{\infty}$ converges in (Y, d_X) for every sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 in (X, d_X), then T is said to be pseudo continuous at x_0. If T is pseudo continuous at every point of (X, d_X), then T is said to be pseudo continuous on (X, d_X).

Every continuous mapping is pseudo continuous. The next example justifies that the converse need not be true.

Example 2: Let $X = Y = \{0\} \cup \left\{\frac{1}{n} : n = 1, 2, 3, \ldots\right\}$ with the usual metric denoted by d_X and d_Y. Define $T: (X, d_X) \to (Y, d_Y)$ by $T\left(\frac{1}{n}\right) = \frac{1}{n}, \forall n = 1, 2, 3, \ldots$ and $T(0) = 1$. Then T is Cauchy continuous on (X, d_X), and T is pseudo continuous on (X, d_X), but T is not continuous on (X, d_X).
Example 2 provides an example for a Cauchy continuous mapping between complete metric spaces, which is not a continuous mapping. Continuity need not imply Cauchy continuity. The next classical example justifies this statement.

Example 3: Let $X = (0, 1)$ be with the usual metric d_X, and let $Y = (1, \infty)$ be with the usual metric d_Y. Define $T: (X, d_X) \to (Y, d_Y)$ by $T(x) = \frac{1}{x}, \forall x \in X$. Then T is continuous on (X, d_X), T is not Cauchy continuous on (X, d_X), and T is not uniformly continuous on (X, d_X).

Proposition 1: Let $T: (X, d_X) \to (Y, d_Y)$ be a pseudo continuous mapping from a complete metric space to a pseudo metric space. Then T is Cauchy continuous on (X, d_X).

Proof: Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence and let it converge to x_0 in the complete metric space (X, d_X). Then the sequence $(Tx_n)_{n=1}^{\infty}$ converges in (Y, d_X) so that it is Cauchy in (Y, d_Y).
It is not possible to expect to generalize Theorem 1 for uniform spaces and for Cauchy nets as it can be seen from the following example.

Example 4: Let $D = \{(i, j) : i = 1, 2, 3, \ldots, j = 1, 2, 3, \ldots\}$. Define $(i, j) \leq (k, l)$ if and only if $i \leq k$, or, $i > k$ and $j \leq l$. This definition provides an order relation in D with respect to which it becomes a directed set. Let $A = \left\{\left(x_{(k,l)}\right)_{(k,l)\in D} : x_{(k,l)} \in \mathbb{R}, \forall (k, l) \in D, Sup_{(k,l)\in D}\left|x_{(k,l)}\right| < \infty\right\}$. For each $(i, j) \in D$, for each non empty finite subset F of D, and for $\left(x_{(k,l)}\right)_{(k,l)\in D}, \left(y_{(k,l)}\right)_{(k,l)\in D} \in A$, let us define $d_{(i,j)}\left(\left(x_{(k,l)}\right)_{(k,l)\in D}, \left(y_{(k,l)}\right)_{(k,l)\in D}\right) = Sup_{(k,l)\leq(i,j)}\left|x_{(k,l)} - y_{(k,l)}\right|$, and define $d_F\left(\left(x_{(k,l)}\right)_{(k,l)\in D}, \left(y_{(k,l)}\right)_{(k,l)\in D}\right) = Sup_{(k,l)\in F}\left|x_{(k,l)} - y_{(k,l)}\right|$. Then each $d_{(i,j)}$ is a pseudo metric on A and each d_F is a pseudo metric on A. To each $(k, l) \in D$, let us define $e_{k,l} = \left(x_{(i,j)}\right)_{(i,j)\in D}$ by $x_{(i,j)} = \frac{1}{j}$ for $(i, j) \leq (k, l)$, and $x_{(i,j)} = 0$ for $(i, j) \nleq (k, l)$. Then the set $X = \{e_{k,l} : (k, l) \in D\}$ is a totally bounded subset in both uniform spaces $\left(A, \left(d_{(i,j)}\right)_{(i,j)\in D}\right)$ and $\left(A, (d_F)_{F,finite}\right)$. Let $T: \left(X, (d_F)_{F,finite}\right) \to \left(X, \left(d_{(i,j)}\right)_{(i,j)\in D}\right)$ be the identity mapping. Then it preserves Cauchy nets, but it is not uniformly continuous on the totally bounded domain, and it is not continuous.

6. SUFFICIENCY OF SEQUENCES

Sequences are sufficient in some cases instead of nets. This is explained in this section.
A uniform space can be given in the form $(X, (d_i)_{i\in D})$, where $(d_i)_{i\in D}$ is the collection of all pseudo metrics on X, which are uniformly continuous with respect to the uniformity of X. Let us write $i \leq j$ in D, when and only when $d_i(x, y) \leq d_j(x, y), \forall x, y \in X$. Then D is a directed set with respect to this order relation. Let us use these notations in this section.

Proposition 2: Let A be a subset of $(X,(d_i)_{i\in D})$. Then a point x_0 in X is in the closure of A if and only if there is a family $\left((x_{i,n})_{n=1}^{\infty}\right)_{i\in D}$ of sequences in X such that $d_i(x_{i,n},x_0)\to 0$ as $n\to\infty$, for every $i\in D$.

Proof: Let A be a subset of X. Then a point x_0 in X is in the closure of A if and only if for every $i\in D$, and for every natural number n, there is an element $x_{i,n}$ in X such that $d_i(x_{i,n},x_0)<\frac{1}{n}$. This provides an argument to complete the proof.

Topologies can be derived by using Kuratowski's closure operators. So, sequences are sufficient to explain concepts in theory of topology, but in a somewhat complicated manner [12,13, 22]. There are many articles [7, 20] for function spaces having Cauchy continuous functions. Let us find the following application for Proposition 2 for a function space.

Let (X,d) be a given metric space. Let A_1 be the collection of all complex valued functions on (X,d) such that their restrictions to each totally bounded subset of (X,d) is bounded. To each totally bounded subset E of (X,d), let us define $p_E(f)=Sup_{x\in E}|f(x)|$, for every $f\in A_1$. Then p_E is a semi norm on the algebra A_1. Here, $d_E(f,g)=p_E(f-g)$ defines the corresponding pseudo metric on A_1. If $\mathcal{D}$ be the collection of all non empty totally bounded subsets of (X,d), then $(A_1,(p_E)_{E\in\mathcal{D}})$ is a complete locally multiplicative convex algebra [18], under natural point wise algebraic operations [15]. Completeness should be verified by means of nets and completeness of the complex field. Let A_2 be the collection of all $f\in A_1$ such that f restricted to each totally bounded subset of (X,d) is uniformly continuous. By Theorem 1, A_2 contains the collection of all Cauchy continuous complex valued functions defined on (X,d).

Proposition 3: With respect to the previous notations, $(A_2,(p_E)_{E\in\mathcal{D}})$ is a complete locally multiplicatively convex algebra.

Proof: Let us first verify that A_2 is closed under algebraic operations. Let $f_1,f_2\in A_2$. Let $E\in\mathcal{D}$. Fix $\varepsilon>0$. Find a $\delta>0$ such that $|f_i(x)-f_i(y)|<\varepsilon$, whenever $d(x,y)<\delta$ with $x,y\in E$, for $i=1,2$. Then $|(f_1+f_2)(x)-(f_1+f_2)(y)|<2\,\varepsilon$, whenever $d(x,y)<\delta$ with $x,y\in E$. For any complex number λ, $|(\lambda f_1)(x)-(\lambda f_2)(y)|\le|\lambda|\varepsilon$, whenever $d(x,y)<\delta$ with $x,y\in E$. Moreover, $|(f_1f_2)(x)-(f_1f_2)(y)|\le|f_1(x)[f_2(x)-f_2(y)]+f_2(y)[f_1(x)-f_1(y)]|\le\left(p_E(f_1)+p_E(f_2)\right)\varepsilon$, whenever $d(x,y)<\delta$ with $x,y\in E$. These inequalities prove that A_2 is a sub algebra of A_1. To prove the proposition, let us verify that A_2 is closed in A_1.
For this purpose, let us consider an element $f\in A_1$, let us consider $E\in\mathcal{D}$, and let us consider a sequence $(f_n)_{n=1}^{\infty}$ in A_2 such that $p_E(f_n-f)\to 0$, as $n\to\infty$. Fix an $\varepsilon>0$. Find an integer n_0 such that $p_E\left(f_{n_0}-f\right)<\frac{\varepsilon}{3}$. Find a $\delta>0$ such that $\left|f_{n_0}(x)-f_{n_0}(y)\right|<\frac{\varepsilon}{3}$, whenever $d(x,y)<\delta$ with $x,y\in E$. Then $|f(x)-f(y)|\le\left|f(x)-f_{n_0}(x)\right|+\left|f_{n_0}(x)-f_{n_0}(y)\right|+\left|f_{n_0}(y)-f(y)\right|<\varepsilon$, whenever $d(x,y)<\delta$ with $x,y\in E$. So, $f\in A_2$. Proposition 2 now implies that A_2 is closed in A_2. This proves the proposition.
Proposition 3 can be extended to uniform spaces, and there is no need to use Proposition 2 in the proof when nets are used. However, by Theorem 1, the condition of uniform continuity on every totally bounded subset can be replaced by Cauchy continuity on the domain in the case of metric spaces in Proposition 3.

Proposition 4: With respect to the previous notations, $(A_2,(p_E)_{E\in\mathcal{D}})$ is a symmetric algebra. This means that for every $f\in A_2$, the function $\left(1+f\bar{f}\right)^{-1}$exists and it is a member of A_2, when $\bar{f}$ denotes the complex conjugate function of f..

Proof: Let us fix $f \in A_2$ and $F \in \mathcal{D}$. Then $1 \leq 1 + f\bar{f} \leq M < \infty$ for some M, on E so that $0 < \frac{1}{M} \leq \left(1 + f\bar{f}\right)^{-1} \leq 1$ on E. This is possible because $1 + f\bar{f}$ is uniformly continuous on E, and $f\bar{f} \geq 0$. Since the inverse map is uniformly continuous on the closed interval $\left[\frac{1}{M}, 1\right]$, the mapping $\left(1 + f\bar{f}\right)^{-1}$ is uniformly continuous on E. This proves that $\left(1 + f\bar{f}\right)^{-1} \in A_2$, whenever $f \in A_2$. This proof is extendable even for uniform spaces X. However, for metric spaces X, it is possible to provide another following proof.

Let us fix $f \in A_2$. Then for any Cauchy sequence $(x_n)_{n=1}^{\infty}$, the sequence $\left(\left(1 + f\bar{f}\right)(x_n)\right)_{n=1}^{\infty}$ is a Cauchy sequence, by Theorem 1. Then $1 \leq \left(1 + f\bar{f}\right)(x_n) \leq M, \forall n$, for some finite M, so that the sequence $\left(\left(1 + f\bar{f}\right)^{-1}(x_n)\right)_{n=1}^{\infty}$ is also a Cauchy sequence. This proves that $\left(1 + f\bar{f}\right)^{-1} \in A_2$, for any $f \in A_2$.

The following corollary is applicable even when X is a uniform space, and it follows from theory of automatic continuity [18].

Corollary 3: Every complex multiplicative linear functional on $(A_2, (p_E)_{E\in\mathcal{D}})$ is sequentially continuous. Every complex multiplicative linear functional on $(A_1, (p_E)_{E\in\mathcal{D}})$ is sequentially continuous.

Proof: One can check that A_1 is also a symmetric algebra.

7. STRONG CONTINUITY

This section is continuity of the previous section, but for uniform spaces [4, 14]. Consider a uniform space in the form $(X, (d_{Xi})_{i\in I})$, where $(d_{Xi})_{i\in I}$ is the collection of all pseudo metrics which are uniformly continuous on the uniform space X. Similarly, let us also consider a uniform space in the form $\left(Y, \left(d_{Yj}\right)_{j\in J}\right)$, where $\left(d_{Yj}\right)_{j\in J}$ is the collection of all pseudo metrics which are uniformly continuous on the uniform space Y.

Let us recall that a function $T: (X, (d_{Xi})_{i\in I}) \to \left(Y, \left(d_{Yj}\right)_{j\in J}\right)$ is said to be uniformly continuous on X, if for given $j \in J$ and for given $\varepsilon > 0$, there is an $i = i(j) \in I$ and there is a $\delta = \delta(\varepsilon, j) > 0$ such that $d_{Yj}(Tx_1, Tx_2) < \varepsilon$ whenever $d_{Xi}(x_1, x_2) < \delta$ in X. This definition is based on a metrization theorem for uniform spaces. This definition provides a new definition meant for a special type of continuous functions.

Definition 2: Let us consider a function $T: (X, (d_{Xi})_{i\in I}) \to \left(Y, \left(d_{Yj}\right)_{j\in J}\right)$. Let $x_0 \in X$. Then T is said to be strongly continuous at x_0, if for given $j \in J$ and for given $\varepsilon > 0$, there is an $i = i(j, x_0) \in I$ and there is a $\delta = \delta(\varepsilon, j, x_0) > 0$ such that $d_{Yj}(Tx_0, Tx) < \varepsilon$ whenever $d_{Xi}(x_0, x) < \delta$ in X. Here i does not depend on ε. However, when "$i = i(j)$", then let us replace "strongly continuous" by "strictly strongly continuous" to frame a new definition. If T is (strictly) strongly continuous at every point of X, then let us say that T is (strictly) strongly continuous on X. Let us use the phrases "strong continuity" and "strict strong continuity" for noun purposes.

Let us observe that uniform continuity implies strict strong continuity on X. Let us further observe that strict strong continuity implies strong continuity, and strong continuity implies continuity.

Proposition 5: With reference to the previous Definition 2, if $(X, (d_{Xi})_{i\in I})$ is metrizable, then continuity implies (strict) strong continuity.

Proof: Let d be a metric on X such that the uniform spaces (X, d) and $(X, (d_{Xi})_{i\in I})$ are equal. Let us fix $x_0 \in X, \varepsilon > 0$, and $j \in J$. Then there is an $i = i(j, x_0) \in I$ and there is a $\delta = \delta(\varepsilon, j, x_0) > 0$ such that $d_{Yj}(Tx_0, Tx) < \varepsilon$ whenever $d_{Xi}(x_0, x) < \delta$ in X. Then there is a $\delta_0 > 0$ such that $d_{Xi}(x_0, x) < \delta$ whenever $d(x_0, x) < \delta_0$ in X. So, $d_{Yj}(Tx_0, Tx) < \varepsilon$ whenever $d(x_0, x) < \delta_0$ in X. This proves the proposition.

The phrase "for given $j \in J$ … there is an $i = i(j) \in I$" in the definition for uniform continuity, and the phrase "for given $j \in J$…there is an $i = i(j, x_0) \in I$" in the definition for strong continuity, consider a function from J to I. Let us denote these associated functions by $A: J \to I$ so that $i = i(j)$ can be replaced by $i = A(j)$. However, let us use the notation $i(j)$ instead of $A(j)$ in the followings so that comparisons can be made with previous notions. The next Proposition 6 can be proved by using a standard argument.

Proposition 6: Consider a function $T: (X, (d_{Xi})_{i\in I}) \to \left(Y, \left(d_{Yj}\right)_{j\in J}\right)$. Let $x_0 \in X$. Let us consider a common function $A: J \to I$ in both following statements. The following two statements are equivalent.

(1) T is strongly continuous at x_0.

(2) For every $j \in J$, $d_{Yj}\left(Tx_{i(j),n}, Tx_0\right) \to 0$ as $n \to \infty$, for any sequence $\left(x_{i(j),n}\right)_{n=1}^{\infty}$ in X for which $d_{Xi(j)}\left(x_{i(j),n}, x_0\right) \to 0$ as $n \to \infty$.

Definition 3: A subset E of $(X, (d_{Xi})_{i\in I})$ is said to be d_{Xi}-totally bounded for a fixed $i \in I$, if for given $\varepsilon > 0$, there are finitely many points $x_1, x_2, \dots, x_n$ in X (or, equivalently in E) such that for every $x \in E$, there is an x_k such that $d_{Xi}(x_k, x) < \varepsilon$.

Definition 4: A sequence $(x_n)_{n=1}^{\infty}$ in $(X, (d_{Xi})_{i\in I})$ is said to be d_{Xi}- Cauchy for a fixed $i \in I$, if for given $\varepsilon > 0$, there an integer n_0 such that $d_{Xi}(x_n, x_m) < \varepsilon, \forall n, m \geq n_0$.

Definition 5: Consider a function $T: (X, (d_{Xi})_{i\in I}) \to \left(Y, \left(d_{Yj}\right)_{j\in J}\right)$. Let us also consider an associated function $A: J \to I$. Let E be a subset of $(X, (d_{Xi})_{i\in I})$. The function T is said to be d_{Yj}- uniformly continuous on E for a fixed $j \in J$, if for given $\varepsilon > 0$, there is a $\delta > 0$ such that $d_{Yj}(Tx_1, Tx_2) < \varepsilon$, whenever $d_{Xi(j)}(x_1, x_2) < \delta$ with $x_1, x_2 \in E$.

Theorem 1 is rewritten in the next Proposition 7.

Proposition 7: Consider a function $T: (X, (d_{Xi})_{i\in I}) \to \left(Y, \left(d_{Yj}\right)_{j\in J}\right)$. Let us consider a common function $A: J \to I$ in the following three statements. Fix $j \in J$ and $i(j)$. The following three statements are equivalent.

(1) The sequence $\left(Tx_{i(j),n}\right)_{n=1}^{\infty}$ is a d_{Yj}- Cauchy sequence in the uniform space Y, for any $d_{Xi(j)}$- Cauchy sequence $\left(x_{i(j),n}\right)_{n=1}^{\infty}$ in the uniform space Y.

(2) The function T is d_{Yj}- uniformly continuous on any $d_{Xi(j)}$- totally bounded subset of X.

(3) The function T is d_{Yj}- uniformly continuous on the set $\left\{x_{i(j),n}: n = 1, 2, 3, \dots\right\}$, for any $d_{Xi(j)}$- Cauchy sequence $\left(x_{i(j),n}\right)_{n=1}^{\infty}$ in X.

Let us end this section with an observation that the collections of strong continuous complex valued functions have algebraic structure. For example, if $f_1: (X, (d_{Xi})_{i\in I}) \to \mathbb{C}$ and $f_2: (X, (d_{Xi})_{i\in I}) \to \mathbb{C}$ are strongly continuous at some point $x_0 \in X$, then $(f_1 + f_2): (X, (d_{Xi})_{i\in I}) \to \mathbb{C}$, $(\lambda f_1): (X, (d_{Xi})_{i\in I}) \to \mathbb{C}$, and $(f_1 f_2): (X, (d_{Xi})_{i\in I}) \to \mathbb{C}$ are also strongly continuous at $x_0 \in X$, where $\lambda \in \mathbb{C}$. One may need an inequality of the type $|f_1(x)| < |f_1(x_0)| + \varepsilon$, whenever $|f_1(x) - f_1(x_0)| < \varepsilon$.

8. BOUNDED MAPPINGS

Let us now come back to metric spaces from uniform spaces. This section is related to Theorem 1. However, this is placed at the end, because bounded mappings are not usually discussed on metric spaces. Bounded linear mappings are known on topological vector spaces [16]. A similar definition can be proposed for non linear bounded mappings on metric spaces.

Definition 6: A subset E of a pseudo metric space (X, d_X) is called a bounded set, when $Sup\,\{d_X(x, y): x, y \in E\}$ is finite. A mapping $T: (X, d_X) \to (Y, d_Y)$ between two pseudo metric spaces is called a bounded mapping when $T(E)$ is a bounded set in (Y, d_Y), for every bounded subset E of (X, d_X).

Proposition 8: Suppose every bounded subset of a pseudo metric space (X, d_X) is totally bounded, and let (Y, d_Y) be a pseudo metric space. Suppose, $T: (X, d_X) \to (Y, d_Y)$ be a mapping such that its restriction to every totally bounded subset of (X, d_X) is uniformly continuous. Then $T: (X, d_X) \to (Y, d_Y)$ is a bounded mapping.

Proof: Consider a bounded subset E of (X, d_X). Then E is totally bounded and hence the restriction of T to E is uniformly continuous so that $T(E)$ is totally bounded. This implies that $T(E)$ is a bounded set and hence $T: (X, d_X) \to (Y, d_Y)$ is a bounded mapping.

Corollary 4: Suppose every bounded subset of a pseudo metric space (X, d_X) is totally bounded, and let (Y, d_Y) be a pseudo metric space. Suppose, $T: (X, d_X) \to (Y, d_Y)$ be a Cauchy continuous mapping. Then $T: (X, d_X) \to (Y, d_Y)$ is a bounded mapping.

Proof: It follows from Theorem 1.

Example 5: Every bounded subset of any finite dimensional Euclidean space with the usual Euclidean metric is totally bounded.

Corollary 5: Let $T: (X, d_X) \to (Y, d_Y)$ be a linear mapping between two metrizable topological vector spaces over the complex field or the real field, when d_X and d_Y are addition invariant metrics. Then the following three statements are equivalent.

(1) According to the definition in functional analysis, $T: (X, d_X) \to (Y, d_Y)$ is a bounded linear mapping.
(2) The mapping $T: (X, d_X) \to (Y, d_Y)$ is continuous on X.
(3) The mapping $T: (X, d_X) \to (Y, d_Y)$ is Cauchy continuous on X.

Proof: (1) $\Leftrightarrow$ (2) is true from functional analysis. (2) implies that $T: (X, d_X) \to (Y, d_Y)$ is uniformly continuous on X, and hence it is Cauchy continuous on X, by Theorem 1. Thus, (2) $\Rightarrow$ (3) is also true. Suppose now that (3) is true. Let $(x_n)_{n=1}^{\infty}$ be a sequence converging to zero in (X, d_X). Let $E = \{0\} \cup \{x_n: n = 1, 2, 3, \dots\}$. By Theorem 1, T restricted to E is uniformly continuous and hence $Tx_n \to 0$, as $n \to \infty$. This proves (2). Thus, (3) $\Rightarrow$ (2) is also true. This proves the corollary.

It is also possible to derive (2) $\Leftrightarrow$ (3) just by using functional analysis arguments without using Theorem 1, in the following way. Suppose (2) is true. Then there is a continuous extension $T_c: (X_c, d_X) \to (Y_c, d_Y)$ of T between the completions. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in (X, d_X). Let it converge to some x_0 in (X_c, d_X). Then $(Tx_n)_{n=1}^{\infty}$ converges to $T_c x_0$ in (Y_c, d_Y). So, $(Tx_n)_{n=1}^{\infty}$ is a Cauchy sequence in (Y, d_Y). This proves that (2) $\Rightarrow$ (3). Suppose now that (3) is true. Let $(x_n)_{n=1}^{\infty}$ be a sequence converging to zero in (X, d_X). Then there is a sequence $(\gamma_n)_{n=1}^{\infty}$ of positive scalars such that $\gamma_n \to \infty$ and such that $\gamma_n x_n \to 0$ as $n \to \infty$. Then $\left(T(\gamma_n x_n)\right)_{n=1}^{\infty}$ is a Cauchy sequence so that it is a bounded sequence, and hence the sequence $\left(\gamma_n^{-1} T(\gamma_n x_n)\right)_{n=1}^{\infty}$ converges to zero in (Y, d_Y). That is, $(Tx_n)_{n=1}^{\infty}$ converges to zero in (Y, d_Y). This proves that (3) $\Rightarrow$ (2).

9. CONCLUSION

A characterization for Cauchy continuity of functions on metric spaces has been found. A definition for Cauchy continuity for functions on general uniform spaces has yet to be proposed so that a characterization can be found.

Conflicts of Interests

The author declared no conflicts of interest.

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