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Adaptive Hybrid Quadrature Scheme for Efficient Numerical Integration in Black-Scholes Option Pricing

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ABSTRACT

Although the Black-Scholes model provides a closed-form analytical solution for European option pricing, numerical methods remain essential when discrete dividend adjustments and integral based pricing formulations are involved. This study investigates the application of the Adaptive Hybrid Quadrature Scheme (AHQS) to the valuation of European call and put options under the Black-Scholes framework with discrete dividend payments. The method combines Simpson's 1/3 rule and two-point Gauss-Legendre quadrature within an adaptive integration framework, where local error estimates are obtained from the difference between the two numerical approximations. Numerical experiments were conducted using empirical market data consisting of five strike prices representing in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) option contracts. The performance of AHQS was evaluated against the composite Simpson's 1/3 rule with $N = 200$ subintervals using absolute error (AE), absolute percentage error (APE), and the number of function evaluations. For call options, AHQS reduced the average number of function evaluations by 61.7%, while the average differences in AE and APE were only +0.0019 and −0.0129, respectively. For put options, the average reduction in function evaluations reached 59.7%, with average differences in AE and APE of −0.0007 and +0.0010, respectively. These results indicate that the changes in pricing accuracy are negligible despite a substantial reduction in computational effort. Overall, the findings demonstrate that AHQS successfully preserves the accuracy of the composite Simpson method while significantly improving computational efficiency. Therefore, AHQS represents a promising deterministic alternative for integral-based European option pricing under the Black–Scholes model with discrete dividends.

***Keywords*:** Adaptive Hybrid Quadrature Scheme, Blac-Scholes Model, European Option Pricing, Numerical Integration, Simpson's Rule.

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1. INTRODUCTION

Numerical integration plays a fundamental role in applied mathematics, particularly in the solution of complex problems for which analytical solutions are unavailable. In quantitative finance, its importance is evident in the valuation of derivative securities, which are often formulated in terms of expected payoffs. Through the Feynman-Kac theorem, these expectations can be linked to the solutions of partial differential equations, highlighting the need for efficient and accurate numerical techniques [1]. One of the most widely used derivative instruments is the option, a financial contract that grants its holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price within a specified period [2]. The development of modern option pricing theory was largely driven by the Black–Scholes model, introduced in 1973, which provides a closed-form solution for European options under the assumptions that the underlying asset follows a geometric Brownian motion and that volatility remains constant [3]. Beyond its mathematical significance in derivative valuation, the Black-Scholes model has also played a pivotal role in shaping the infrastructure and practices of modern financial markets [4].

Despite its analytical tractability, the Black-Scholes model has several limitations in capturing real market conditions. Assumptions such as constant volatility and a fixed risk-free interest rate are often inconsistent with empirical observations, leading to discrepancies between theoretical and market prices [5]. These limitations have motivated the development of various extensions to the original framework, including stochastic volatility models, jump processes, and fractional approaches designed to capture more complex market dynamics [3]. Nevertheless, numerical methods remain indispensable in many practical applications, particularly when dividends or time-varying parameters are incorporated into the pricing framework. Over the years, numerical techniques such as finite difference methods and Monte Carlo simulation have been widely employed to solve Black-Scholes type problems [6]. In addition, finite volume methods have been developed for generalized Black–Scholes equations in multi-asset settings and have demonstrated promising numerical performance [7]. Numerical integration approaches, such as Simpson's rule, have also been adopted as efficient deterministic alternatives for option valuation [8]. Furthermore, higher-order implementations of Simpson's rule have been successfully applied to a variety of numerical problems, demonstrating improvements in both integration accuracy and computational efficiency [9].

Classical numerical integration methods such as Simpson's rule possess fourth-order accuracy, however, in practice they often face challenges in balancing numerical accuracy and computational cost, particularly when dealing with functions that exhibit varying local behavior [8]. To address these limitations, a variety of hybrid numerical approaches have been developed to enhance computational performance.

One such approach combines Simpson's rule with Hermite interpolation, achieving simultaneous improvements in both efficiency and accuracy [10]. More recently, adaptive and hybrid integration strategies have been introduced to further improve computational efficiency through the dynamic allocation of computational resources according to the local characteristics of the integrand. A notable example is the integration of Simpson's rule with Gauss quadrature within an adaptive framework, which has been shown to improve computational efficiency without sacrificing numerical accuracy [11]. Gauss quadrature techniques have also been successfully applied to optical wavefront reconstruction problems, demonstrating strong performance in the numerical evaluation of two-dimensional integrals [12].

In addition, Gauss-Legendre quadrature has proven effective in improving estimation accuracy in time-series-based numerical integral models [13]. Motivated by these developments, this study investigates the application of the Adaptive Hybrid Quadrature Scheme (AHQS) to the integral formulation of the Black-Scholes model. The primary objective is to evaluate whether the adaptive combination of Simpson's rule and Gauss-Legendre quadrature can improve computational efficiency while maintaining a level of pricing accuracy comparable to that of conventional numerical integration methods for European option valuation.

2. MATERIALS AND METHODS

This section outlines the theoretical background and methodology used in this study. The discussion covers the composite Simpson's 1/3 rule, Gauss-Legendre quadrature, the Adaptive Hybrid Quadrature Scheme (AHQS), the Black-Scholes model, the integral formulation of option pricing, and the numerical procedures employed in the simulations.

2.1. Composite Simpson's 1/3 Rule

The composite Simpson's 1/3 rule is a numerical integration method that approximates the value of a definite integral using quadratic polynomial interpolation over each subinterval. Let the interval $[a, b]$ be partitioned into n equally spaced subintervals, where n is an even integer and $h = \frac{b-a}{n}$. The composite Simpson's 1/3 approximation is given by

$$S(a,b) = \frac{h}{3}\left[f(a) + f(b) + 4\sum_{k=1}^{n/2} f\,(x_{2k-1}) + 2\sum_{k=1}^{n/2-1} f\,(x_{2k})\right].$$

The truncation error associated with Simpson's rule is expressed as

$$E_S = I - S(a,b) = -\frac{(b-a)h^4}{180} f^{(4)}(\xi), \qquad \xi \in [a,b].$$

This result indicates that Simpson's rule possesses fourth-order convergence, providing a high level of accuracy for sufficiently smooth functions [14].

2.2. Gauss-Legendre Quadrature

Gauss-Legendre quadrature is a numerical integration technique that approximates a definite integral by selecting evaluation points and weights in an optimal manner. For an interval $[a, b]$, the Gauss-Legendre quadrature formula is expressed as

$$G(a,b) = \frac{b-a}{2}\sum_{i=1}^{m} w_i\, f\left(\frac{b-a}{2} t_i + \frac{a+b}{2}\right),$$

where t_i denotes the roots of the Legendre polynomial $P_m(t)$, and w_i represents the corresponding quadrature weights. The quadrature weights are given by

$$w_i = \frac{2}{(1-t_i^2)[P_m'(t_i)]^2}.$$

In the case of two-point Gauss-Legendre quadrature, the nodes and weights are

$$t_{1,2} = \pm\frac{1}{\sqrt{3}}, \qquad w_{1,2} = 1.$$

The truncation error associated with the two-point Gauss-Legendre rule is

$$E_G = I - G(a,b) = \frac{(b-a)^5}{4320} f^{(4)}(\eta), \qquad \eta \in [a,b].$$

This method also achieves fourth-order accuracy while requiring only two function evaluations per subinterval, making it computationally efficient for numerical integration problems [15].

2.3. Adaptive Hybrid Quadrature Scheme (AHQS)

The Adaptive Hybrid Quadrature Scheme (AHQS) is a numerical integration approach that combines the composite Simpson's 1/3 rule and Gauss-Legendre quadrature within an adaptive framework to improve both accuracy and computational efficiency. For each subinterval, the local error estimate is computed from the difference between the Simpson and Gauss approximations:

$$E = |S - G|,$$

where S denotes the approximation obtained from the composite Simpson's 1/3 rule and G represents the approximation obtained from Gauss-Legendre quadrature.

If the estimated error satisfies the prescribed tolerance, the Gauss approximation is accepted as the numerical solution on that subinterval. Otherwise, the interval is recursively subdivided until the desired accuracy criterion is met. Through this adaptive refinement strategy, computational effort is concentrated on regions where the integrand exhibits more complex behavior, while fewer evaluations are performed in smoother regions.

By combining the robustness of Simpson's rule with the efficiency of Gauss quadrature, AHQS is capable of producing accurate integral approximations at a lower computational cost than conventional fixed-partition numerical integration methods [11].

2.4. Black-Scholes Model

The Black-Scholes model is a mathematical framework widely used for determining the fair value of European-style options. The model assumes that the underlying asset price follows a geometric Brownian motion process, while both volatility and the risk-free interest rate remain constant over time.

Under these assumptions, the prices of European call and put options without dividends are respectively given by

$$C = S_0 N(d_1) - K e^{-r(T-t)} N(d_2),$$

and

$$P = K e^{-r(T-t)} N(-d_2) - S_0 N(-d_1),$$

where

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Here, S_0 denotes the current asset price, K is the strike price, r represents the risk-free interest rate, σ is the volatility of the underlying asset, and $T-t$ denotes the remaining time to maturity. The function $N(\cdot)$ represents the cumulative distribution function of the standard normal distribution.

To account for a single discrete dividend payment D occurring at time t_d, the asset price is adjusted according to

$$S^* = S_0 - De^{-rt_d}.$$

The adjusted asset price S^* is then substituted into the Black-Scholes pricing formulas to obtain the corresponding option values [8].

2.5. Integral Formulation of Option Pricing

In the numerical integration approach, the price of a European option can be expressed as the discounted expected payoff under a lognormal probability density function. The price of a European call option is given by

$$C(S^*, K, T, r, \sigma) = e^{-r(T-t)} \int_K^\infty (S_T - K)\, f(S_T)\, dS_T,$$

while the price of a European put option is defined as

$$P(S^*, K, T, r, \sigma) = e^{-r(T-t)} \int_0^K (K - S_T)\, f(S_T)\, dS_T.$$

The lognormal probability density function of the terminal asset price S_T is given by

$$f(S_T) = \frac{1}{S_T \sigma\sqrt{2\pi(T-t)}} \exp\left(-\frac{\left[\ln\left(\frac{S_T}{S^*}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]^2}{2\sigma^2(T-t)}\right).$$

Since the call option integral has an infinite upper bound, numerical integration is performed over the truncated interval $[K, S_{\max}]$, where $S_{\max}$ is chosen sufficiently large so that the truncation error becomes negligible. In contrast, the put option integral is evaluated over the finite interval $[0, K]$. These integral representations form the basis of the numerical pricing procedure and are subsequently solved using the Adaptive Hybrid Quadrature Scheme (AHQS) [8].

2.6. Research Methodology

This study adopts an applied quantitative approach to evaluate the performance of the Adaptive Hybrid Quadrature Scheme (AHQS) in pricing European options under the Black-Scholes framework with discrete dividend adjustments. The numerical procedure is based on approximating the integral representation of option prices using an adaptive combination of the composite Simpson's 1/3 rule and Gauss-Legendre quadrature.

The implementation procedure employed in this study can be summarized as follows:

- Specify the Black-Scholes model parameters, including S_0, K, r, σ, T, D, and t_d.
- Compute the dividend-adjusted asset price using
$$S^* = S_0 - De^{-rt_d}.$$
- Construct the integral formulations for European call and put option prices based on the lognormal probability density function.
- Define the integration interval and the numerical error tolerance.
- Compute numerical approximations of the integral using the composite Simpson's 1/3 rule and Gauss-Legendre quadrature.
- Estimate the local integration error using
$$E = |S - G|.$$
- Accept the approximation if the estimated error satisfies the prescribed tolerance; otherwise, recursively subdivide the interval until the accuracy criterion is achieved.
- Evaluate the option price from the resulting numerical integration.

All numerical simulations were implemented in Python to assess the accuracy and computational efficiency of the Adaptive Hybrid Quadrature Scheme for European option pricing.

3. RESULTS AND DISCUSSION

This section presents the results of implementing the Adaptive Hybrid Quadrature Scheme (AHQS) for European option pricing under the Black-Scholes model with discrete dividend adjustments. The numerical performance of AHQS is evaluated by comparing its pricing accuracy and computational efficiency with those of the composite Simpson's 1/3 rule. The comparison is conducted using empirical option data representing in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) contracts. The analysis focuses on option prices, absolute error (AE), absolute percentage error (APE), and the number of function evaluations required by each numerical integration method.

3.1. Simulation Parameters

The simulation parameters used in this study were adopted from the empirical dataset reported in [8]. In that study, stock price data were obtained from Yahoo Finance, while risk-free interest rate data were collected from the Federal Reserve Economic Data (FRED) database maintained by the Federal Reserve Bank of St. Louis. Employing the same dataset enables a consistent comparison between the composite Simpson's 1/3 rule and the Adaptive Hybrid Quadrature Scheme (AHQS) under identical numerical conditions.

Five strike prices were selected for both call and put options to represent different moneyness categories, namely in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) contracts. The corresponding market prices used as benchmarks in the numerical experiments are presented in Tables 1 and 2.

Table 1. Market Data for Call Option.

Zone	Strike Price (K)	Market Price	Moneyness
ITM	140	51.40	In-The-Money
ITM	155	38.85	In-The-Money
ATM	190	14.67	At-The-Money
OTM	220	4.60	Out-Of-The-Money
OTM	240	1.89	Out-Of-The-Money

Table 2. Market Data for Put Option.

Zone	Strike Price (K)	Market Price	Moneyness
OTM	140	1.40	Out-Of-The-Money
OTM	155	3.11	Out-Of-The-Money
ATM	190	14.00	At-The-Money
ITM	220	34.10	In-The-Money
ITM	240	51.40	In-The-Money

The selected option contracts cover a broad range of moneyness conditions, allowing the numerical methods to be evaluated under different pricing scenarios. This classification is important because the accuracy and computational behavior of numerical integration methods may vary depending on the relative position of the strike price with respect to the underlying asset price.

In addition to the option market data, the Black-Scholes model parameters used in the numerical simulations are presented in Table 3. These parameters serve as inputs to the integral pricing formulation and are subsequently employed in the implementation of the Adaptive Hybrid Quadrature Scheme (AHQS).

Table 3. Black-Scholes Model Parameters.

Parameter	Value	Description
S_0	189.11	Initial asset price
S^*	189.10	Dividend-adjusted asset price
r	3.46%	Risk-free interest rate
σ	49.91%	Annualized volatility
$T-t$	0.1972	Time to maturity
t_d	0.1753	Dividend payment time
D	0.01	Discrete dividend
N	200	Number of numerical subintervals

In the numerical implementation, the call option integral, which possesses an infinite upper bound, is truncated to the finite interval $[K, S_{max}]$. The value of $S_{\max}$ is chosen sufficiently large to ensure that the truncation error remains negligible while maintaining computational stability. In contrast, the put option integral is evaluated over the finite interval $[0, K]$, which naturally corresponds to the payoff domain of a European put option.

The resulting numerical approximations obtained from the composite Simpson's 1/3 rule and AHQS are then compared in terms of option prices, absolute error (AE), absolute percentage error (APE), and the number of function evaluations.

3.2. Analysis of Option Pricing Results

The performance of the Adaptive Hybrid Quadrature Scheme (AHQS) was evaluated against the composite Simpson's 1/3 rule with $N = 200$ subintervals for the pricing of European call and put options. The comparison was conducted under an equivalent accuracy basis using several strike prices representing different moneyness categories, namely in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) contracts.

The evaluation metrics considered in this study include the approximated option price (Price), absolute error (AE), absolute percentage error (APE), and the number of function evaluations (Ev). In the reported results, the notation S refers to the composite Simpson's 1/3 rule, whereas A denotes the Adaptive Hybrid Quadrature Scheme (AHQS). The percentage reduction in function evaluations is measured using

$$\Delta Eval\% = \frac{S.Ev - A.Ev}{S.Ev} \times 100\%.$$

A positive value of $\Delta Eval\%$ indicates that AHQS requires fewer function evaluations and is therefore more computationally efficient than the composite Simpson method.

In addition, the differences in absolute error and absolute percentage error are defined as

$$\Delta AE = A.AE - S.AE,$$

and

$$\Delta APE = A.APE - S.APE.$$

Positive values of ΔAE or ΔAPE indicate that AHQS produces slightly larger errors than the composite Simpson method, whereas negative values indicate that AHQS achieves lower errors.

Tables 4 and 5 summarize the numerical results obtained for call and put options, respectively. The comparison focuses on the trade-off between pricing accuracy and computational efficiency. In particular, the analysis examines whether the adaptive allocation strategy employed by AHQS can reduce the number of function evaluations while maintaining an accuracy level comparable to that of the composite Simpson's 1/3 rule.

Table 4. Comparison of European Call Option Pricing Results Obtained Using the Composite Simpson's 1/3 Rule and the Adaptive Hybrid Quadrature Scheme (AHQS).

Zone	K	Market Price	S.Price	S.AE	S.APE (%)	S.Ev
ITM	140	51.4000	51.4025	0.0025	0.0049	201
ITM	155	38.8500	38.7490	0.1010	0.2600	201
ATM	190	14.6700	16.8682	2.1982	14.9841	201
OTM	220	4.6000	6.9542	2.3542	51.1782	201
OTM	240	1.8900	3.5856	1.6956	89.7122	201
Average	–	–	–	**1.2703**	**31.2279**	**201**

Zone	K	A.Price	A.AE	A.APE (%)	A.Ev	ΔEval (%)	ΔAE	ΔAPE
ITM	140	51.3964	0.0036	0.0071	75	+62.7	+0.0011	+0.0021
ITM	155	38.7354	0.1146	0.2950	75	+62.7	+0.0136	+0.0350
ATM	190	16.8655	2.1955	14.9662	75	+62.7	-0.0026	-0.0179
OTM	220	6.9521	2.3521	51.1329	75	+62.7	-0.0021	-0.0454
OTM	240	3.5848	1.6948	89.6738	85	+57.7	-0.0007	-0.0383
Average	–	–	**1.2721**	**31.2150**	**77**	**+61.7**	**+0.0019**	**-0.0129**

Table 5. Comparison of European Put Option Pricing Results Obtained Using the Composite Simpson's 1/3 Rule and the Adaptive Hybrid Quadrature Scheme (AHQS).

Zone	K	Market Price	S.Price	S.AE	S.APE (%)	S.Ev
OTM	140	1.4000	1.3503	0.0497	3.5474	201
OTM	155	3.1100	3.5948	0.4848	15.5879	201
ATM	190	14.0000	16.4759	2.4759	17.6852	201
ITM	220	34.1000	36.3579	2.2579	6.6215	201
ITM	240	51.4000	52.8533	1.4533	2.8274	201
Average	–	–	–	**1.3443**	**9.2539**	**201**

Zone	K	A.Price	A.AE	A.APE (%)	A.Ev	ΔEval (%)	ΔAE	ΔAPE
OTM	140	1.3501	0.0499	3.5633	85	+57.7	+0.0002	+0.0160
OTM	155	3.5946	0.4846	15.5835	95	+52.7	-0.0001	-0.0044
ATM	190	16.4747	2.4747	17.6766	85	+57.7	-0.0012	-0.0086
ITM	220	36.3651	2.2651	6.6425	75	+62.7	+0.0072	+0.0210
ITM	240	52.8435	1.4435	2.8084	65	+67.7	-0.0098	-0.0190
Average	–	–	**1.3436**	**9.2549**	**81**	**+59.7**	**-0.0007**	**+0.0010**

Based on Tables 4 and 5, the Adaptive Hybrid Quadrature Scheme (AHQS) achieves a level of accuracy that is highly comparable to that of the composite Simpson's 1/3 rule with $N = 200$, while requiring substantially fewer function evaluations. For call options, the average number of function evaluations decreases from 201 to 77, corresponding to a computational saving of approximately 61.7%. Similarly, for put options, the average number of function evaluations is reduced from 201 to 81, yielding a computational saving of approximately 59.7%.

From an accuracy perspective, the average absolute error (AE) and absolute percentage error (APE) produced by the two methods are nearly identical. For call options, the average AE changes from 1.2703 to 1.2721, resulting in an average ΔAE of $+0.0019$, while the average APE changes from 31.2279% to 31.2150%, corresponding to an average ΔAPE of -0.0129. For put options, the average AE changes from 1.3443 to 1.3436, yielding an average ΔAE of -0.0007, whereas the average APE changes from 9.2539% to 9.2549%, corresponding to an average ΔAPE of $+0.0010$. These results indicate that the differences in accuracy between the two methods are negligible and can be considered practically insignificant.

The findings demonstrate that the adaptive mechanism employed by AHQS allocates evaluation points more efficiently than the composite Simpson method with a fixed number of subintervals. Although the present study uses the composite Simpson's 1/3 rule with $N = 200$ as the benchmark method, the observed results remain consistent with the findings of Asgedom and Kefela [11], who reported that AHQS can substantially reduce the number of function evaluations without causing a meaningful loss of numerical accuracy. Consequently, AHQS may be regarded as an efficient alternative for integral-based option pricing under the Black-Scholes framework, particularly when a balance between numerical accuracy and computational cost is required.

4. CONCLUSIONS

This study successfully implemented the Adaptive Hybrid Quadrature Scheme (AHQS) for integral-based European option pricing under the Black-Scholes framework with discrete dividend adjustments. The proposed approach combines Simpson's 1/3 rule and two-point Gauss-Legendre quadrature within an adaptive framework to obtain accurate numerical approximations while reducing computational cost. The numerical experiments demonstrate that AHQS maintains a level of accuracy comparable to that of the composite Simpson's 1/3 rule with $N = 200$. For call options, the average ΔAE and ΔAPE were $+0.0019$ and -0.0129, respectively, while for put options the corresponding values were -0.0007 and $+0.0010$. Since all differences remain close to zero, the results indicate that AHQS preserves pricing accuracy to a practically equivalent level despite using substantially fewer function evaluations.

From a computational perspective, AHQS significantly reduced the number of function evaluations required to achieve comparable accuracy. The average number of evaluations decreased from 201 to 77 for call options and from 201 to 81 for put options, corresponding to computational savings of 61.7% and 59.7%, respectively. These results confirm that the adaptive mechanism of AHQS allocates computational effort more efficiently than a fixed-partition integration strategy without introducing a meaningful loss of accuracy. Overall, the findings demonstrate that AHQS provides an effective balance between numerical accuracy and computational efficiency in integral-based European option pricing.

The results further support the central conclusion of Asgedom and Kefela [11] that adaptive hybrid quadrature methods can substantially reduce computational cost while preserving the quality of numerical approximations. Therefore, AHQS may be considered a practical and efficient deterministic alternative for option pricing applications where both accuracy and computational performance are important.

For future research, the performance of AHQS may be further investigated using different numbers of subintervals in the composite Simpson method as benchmarks to examine the relationship between discretization level, numerical accuracy, and computational efficiency. In addition, a broader dataset involving a larger number of option contracts and longer observation periods may provide a more comprehensive assessment of the robustness and practical applicability of AHQS in European option pricing under the Black-Scholes framework.

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