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# A New Three-Step Modification of a Fifth-Order Iterative Method with Weight Functions for Solving Nonlinear Equations

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**ABSTRACT**

Solving nonlinear equations is a fundamental problem in numerical analysis, as analytical solutions are generally unattainable for most practical cases. While multipoint iterative methods incorporating weight functions have proven effective in achieving higher convergence orders without additional derivative evaluations, existing methods such as the fifth-order scheme of Liu et al. (2021) still leave room for improvement in convergence order within the same computational cost. This study proposes a three-step iterative method constructed by modifying Liu's scheme through the introduction of a frozen derivative strategy combined with a linear weight function governed by a free scalar parameter. Convergence analysis via Taylor series expansion proves that the proposed method achieves sixth-order convergence for general parameter values and seventh-order convergence for a specific parameter choice. Numerical simulations on five standard test functions, benchmarked against Liu's fifth-order methods, Bi-Ren-Wu's seventh-order method, and Tao-Wan's seventh-order method, confirm that the proposed scheme converges rapidly, produces highly accurate approximations, and yields computational convergence orders consistent with the theoretical results. These findings establish the proposed method as an efficient and competitive alternative for solving nonlinear equations.

***Keywords***: Nonlinear equations, Weight function, Seventh-order iterative methods, Three-step methods, Convergence order, Numerical analysis.

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## 1. INTRODUCTION

Solving nonlinear equations $f(x) = 0$ is one of the fundamental problems in mathematics, especially in numerical analysis. This is because exact solutions are usually not analytically tractable. Numerical computation becomes the main strategy in tackling such issues. The classical Newton technique is the basis of the order of convergence quadratic for a long time. The desire for more efficient computation, however, has led academics to explore for novel approaches such as multipoint iteration. This strategy has been demonstrated to improve the rate of convergence considerably without requiring computation of higher-order derivatives, such as second or third order derivatives which demand a substantial processing cost (Petković et al., 2013).

In its development, iterative methods have not only been directed toward increasing the order of convergence, but also toward improving the efficiency of the number of function evaluations in each iteration. Multistep methods generally have better performance than one-step methods because they can produce a higher order of convergence with a more flexible iterative structure (Noor et al., 2006; Noor, 2007). In addition, various high-order methods have also been developed to overcome the limitations of classical methods, particularly when higher-order derivatives are difficult to obtain or require substantial computational cost (Nazeer et al., 2016).

In improving the efficiency of computational algorithms, various modifications of iterative methods have been developed by emphasizing the enhancement of convergence order and the reduction of function evaluation costs. Cordero and Torregrosa (2007) developed a variation of an iterative method through a modification of a fifth-order quadrature formula to improve the rate of convergence. Subsequently, the use of weight functions began to be widely applied because it has proven effective in manipulating the error equation, allowing the order of convergence to be increased without adding derivative evaluations. This was demonstrated by Behl et al. (2016) by constructing an optimal family of eighth-order algorithms. This trend has resulted to the weight function approach becoming a leading methodology in the development of flexible and efficient families of iterative methods, notably in the efficiency bound of iterative methods (Ogbereyivwe & Ojo-Orobosa, 2021; Zein, 2023). Moreover, high-order techniques with relatively small number of function evaluations have been established, such as sixteenth-order method with five function evaluations, which shows that the increase in the convergence order may be achieved by an adequate iterative design (Ullah et al., 2013).

Recently, Liu et al. (2021) introduced an efficient family of fifth-order iterative methods by utilizing weight functions. Liu stated that the proposed method requires four function evaluations per iteration to produce an efficiency index ratio of 1.495. Although the method has good computational performance, the use of four function evaluations theoretically still allows a higher convergence order to be achieved, even up to order eight. This indicates that the method of Liu et al. (2021) still has room for further development.

Based on this gap, this study proposes a three-step iterative method modified from the method of Liu et al. (2021) for solving nonlinear equations. The modification is carried out by adding a third step combined with a weight function, so that it is expected to increase the order of convergence to six, and even has the potential to reach order seven, without requiring additional function evaluations. Therefore, this study is directed toward obtaining an iterative method that is more efficient in terms of both convergence order and the number of function evaluations used.

## 2. MATERIALS AND METHODS

In this section, we will introduce various theoretical notions that are necessary in the creation and analysis of iterative approaches for solving nonlinear equations. The section starts with nonlinear equations and then deals with various basic concepts such as efficiency index, error equation, approximated computational order of convergence, geometric series, and Taylor's theorem.

### 2.1. Nonlinear Equations

The general form of a nonlinear equation is given by

$$f(x) = 0.$$

A nonlinear equation may be an algebraic equation, which involves operations such as addition, multiplication, subtraction, division, and radicals, or a transcendental equation, which involves trigonometric, logarithmic, exponential, and other transcendental functions (Hoffman, 2001). The objective in solving a nonlinear equation is to determine a value $r$ such that $f(r) = 0$. In numerical analysis, this value is commonly approximated by constructing an iterative sequence that converges to the root.

### 2.2. Standard Definition

Let $r$ be a simple root of the nonlinear equation $f(x) = 0$. The following definitions and theorem are used to support the analysis of iterative methods.

**Definition 1 (Efficiency Index).** (Gautschi, 2012) Let $\rho$ denote the number of function evaluations required in each iteration and let $p$ denote the order of convergence of an iterative method. The efficiency index of the iterative method is defined by

$$IE = p^{\frac{1}{\rho}}.$$

**Definition 2 (Error Equation).** (Weerakoon and Fernando, 2000) If $e_n = x_n - r$ denotes the error at the $n$-th iteration, where $r$ is the exact root, then the equation

$$e_{n+1} = Ce_n^p + O\left(e_n^{p+1}\right)$$

is called the error equation. The value of $p$ in the error equation represents the order of convergence.

**Definition 3 (Approximated Computational Order of Convergence).** (Sanchez et al., 2010) The approximated computational order of convergence (ACOC) of the iterative sequence $\{x_n\}_{n=1}^{\infty}$ with initial value $x_0$ is defined by

$$\text{ACOC} \approx \frac{\ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right)}{\ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right)}, \qquad n = 2,3, \dots.$$

Computationally, ACOC is used to determine an approximation of the convergence order of a method. The terms $x_{n-2}$, $x_{n-1}$, $x_n$, and $x_{n+1}$ respectively represent iterative approximations to the root.

**Definition 4 (Geometric Series).** (Stewart, 2016) The geometric series

$$\sum_{n=1}^{\infty} a\, s^{n-1} = a + as + as^2 + \cdots$$

is convergent if $|s| < 1$, and its sum is expressed as

$$\sum_{n=1}^{\infty} a\, s^{n-1} = \frac{a}{1-s}, \qquad |s| < 1.$$

If $|s| \geq 1$, then the geometric series is divergent.

**Theorem 1 (Taylor's Theorem).** (Bartle and Sherbert, 2011) Let $n \in \mathbb{N}$, $I = [a,b]$, and $f: I \to \mathbb{R}$ such that $f(x)$ and its derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$ are continuous on $I$, while $f^{(n+1)}(x)$ exists on $(a,b)$. If $x_0 \in I$, then for any $x \in I$, there exists a point $c$ between $x$ and $x_0$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

### 2.3. Liu's Fifth-Order Iterative Method

Liu et al. (2021) presented a family of 5th order iterative methods for solving nonlinear equations in the form $f(x) = 0$. The approach is developed on the assumption that $r \in I$ is a simple zero of a differentiable function $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an open interval. This assumption implies that $f(r) = 0$ and $f'(r) \neq 0$, so the convergence analysis can be carried out through the error term $e_n = x_n - r$. The main objective of the method is to increase the convergence order by using a suitable weight function without involving higher-order derivatives.

The iterative process begins with the Newton approximation

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Using this intermediate value, Liu et al. proposed the following two-step iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - H(\eta_n)\frac{f(y_n)}{f'(x_n)},$$

where

$$\eta_n = \frac{f'(y_n)}{f'(x_n)} - 1.$$

In this formulation, $H$ denotes a weight function. The role of this function is essential because it controls the coefficients appearing in the error expansion. To obtain fifth-order convergence, the weight function must satisfy the following conditions:

$$H(0) = 1, \qquad H'(0) = -1, \qquad H''(0) = \frac{5}{2}.$$

Under these conditions, the fourth-order error term can be eliminated, and the corresponding error equation takes the form

$$e_{n+1} = Ce_n^5 + O(e_n^6),$$

where $C$ is the asymptotic error constant. Hence, the method has fifth-order convergence. This result shows that the weight function approach can improve the convergence behavior of a multipoint iterative method through the manipulation of the error equation.

The computational structure of Liu et al.'s method requires four evaluations per iteration, namely $f(x_n)$, $f'(x_n)$, $f(y_n)$, and $f'(y_n)$. Therefore, its efficiency index is given by

$$IE = 5^{1/4} \approx 1.49535.$$

Although the method achieves fifth-order convergence with relatively efficient computational cost, the use of four function evaluations theoretically still leaves room for further improvement. According to the optimality concept for multipoint methods without memory, four function evaluations can potentially produce a seventh or eighth-order method. Therefore, Liu et al.'s fifth-order scheme provides an important basis for further modification, especially for constructing a new iterative method with a higher convergence order while preserving the same number of function evaluations.

### 2.4. Bi-Ren-Wu's Seventh-Order Iterative Method

Bi et al. (2008) proposed a one-parameter family of seventh-order iterative methods by systematically extending King's fourth-order scheme through a divided difference approximation of the second derivative. The central idea is to replace $f''(y_n)$, which would ordinarily appear in a direct Newton-correction step, with the approximation

$$f''(y_n) \approx 2f[z_n, x_n, x_n] = \frac{2\big(f[z_n, x_n] - f'(x_n)\big)}{z_n - x_n}$$

where $f[z_n, x_n]$ and $f[z_n, x_n, x_n]$ denote the standard divided differences of first and second order, respectively. This substitution makes the third-step correction computable exclusively from function values already evaluated in the preceding steps, avoiding any additional derivative computation. The resulting family of methods is formulated as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \cdot \frac{f(y_n)}{f'(x_n)}$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}$$

where $\beta \in \mathbb{R}$ is a free parameter and $f[\cdot,\cdot]$, $f[\cdot,\cdot,\cdot]$ denote the standard divided differences.

The convergence analysis, conducted under the assumption $f \in C^3(D)$, proceeds via Taylor expansion in powers of $e_n = x_n - x^*$. Setting $c_k = \frac{f^{(k)}}{k!\, f'(x^*)}$, the analysis yields the following seventh-order error equation:

$$e_{n+1} = 2c_2^2\, c_3(c_3 - (1 + 2\beta)c_2^2)e_n^7 + O(e_n^8).$$

A notable feature of this family is that the asymptotic error constant depends nontrivially on $\beta$, so that distinct values of the parameter yield genuinely distinct methods, each with a different error behavior. This distinguishes the family of Bi et al. from certain families in the literature whose error constants are formally parameter dependent but effectively invariant a distinction explicitly noted by the authors in their convergence analysis.

Regarding computational cost, each iteration requires three evaluations of $f$, namely $f(x_n)$, $f(y_n)$, $f(z_n)$, together with one evaluation of $f'(x_n)$. With all evaluations counted at equal cost, the efficiency index is

$$\mathrm{IE} = 7^{\frac{1}{4}} \approx 1.627$$

which exceeds that of Newton's method $\left(\mathrm{IE} = 2^{\frac{1}{2}} \approx 1.414\right)$, King's family $\left(\mathrm{IE} = 4^{\frac{1}{3}} \approx 1.587\right)$, and standard sixth-order four-evaluation methods $\left(\mathrm{IE} = 6^{\frac{1}{4}} \approx 1.565\right)$.

### 2.5. Tao-Wan's Seventh-Order Iterative Method

Tao and Wan (2020) developed a Newton-type iterative scheme with seventh-order convergence by extending King's fourth-order method through the introduction of a weight function in the third step. The method is constructed under the assumption that $f: I \subset \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function on an open interval $I$, with $a \in I$ being a simple root of $f(x) = 0$. The iterative process proceeds in three steps, where the final correction is governed by a bivariate weight function $h(u_k, v_k)$ depending on the ratios

$$u_k = \frac{f(y_k)}{f(x_k)}, \qquad v_k = \frac{f(z_k)}{f(y_k)}.$$

The complete iterative scheme is given by

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$z_k = y_k - \frac{f(x_k)\, f(y_k)}{\big(f(x_k) - 2f(y_k)\big)f'(x_k)}$$

$$x_{k+1} = z_k - \frac{f(x_k)\, f(z_k)}{\big(f(x_k) - 2f(y_k)\big)\, f'(x_k)} \cdot h(u_k, v_k)$$

Using the standard Taylor expansion at $f(a) = 0$ and the notation $c_n = \frac{\frac{1}{n!}f^{(n)}}{f'(a)}$, Tao and Wan established that seventh-order convergence is attained if and only if the weight function $h$ satisfies the following conditions:

$$h(0,0) = 1, \qquad h'_u(0,0) = 0, \qquad h'_v(0,0) = 1$$

$$h''_{uu}(0,0) = 2, \qquad |h''_{uv}|(0,0) < \infty, \qquad |h''_{vv}|(0,0) < \infty$$

$$h''_{uu}(0,0) = 2, \qquad |h''_{uv}|(0,0) < \infty, \qquad |h''_{vv}|(0,0) < \infty$$

Under these constraints, the general error equation takes the form

$$e_{k+1} = -c_2^2(c_2^2 - c_3)\left[c_2^2\left(-4 + h''_{uv}(0,0)\right) - c_3\left(-2 + h''_{uv}(0,0)\right)\right]e_k^7 + O(e_k^8).$$

A concrete realization satisfying the above conditions is obtained by selecting

$$h(u_k, v_k) = 1 + \frac{f(z_k)}{f(x_k)} + \frac{f(z_k)}{f(y_k)} + \frac{f^2(y_k)}{f^2(x_k)} + \frac{f^2(z_k)}{f^2(y_k)}$$

for which the error equation simplifies to

$$e_{k+1} = -c_2^2(c_2^2 - c_3)(-3c_2^2 + c_3)e_k^7 + O(e_k^8).$$

Each iteration of this scheme requires one evaluation of $f'(x_k)$ and three evaluations of $f$, namely $f(x_k)$, $f(y_k)$, and $f(z_k)$, resulting in four function evaluations per iteration. The corresponding efficiency index is therefore

$$\mathrm{IE} = 7^{\frac{1}{4}} \approx 1.627.$$

Numerical experiments reported by Tao and Wan confirm that the method consistently outperforms Newton's method, King's family, and several sixth-order schemes across a range of test functions.

## 3. RESULTS AND DISCUSSIONS

To accelerate the convergence rate of the fifth-order two-step iterative method without increasing the computational burden of new derivative evaluations, the modification in this paper is the introduction of the third step. In constructing this third step, we utilize the frozen derivative approach. This approach evaluates the function at the new point $z_n$, which is $f(z_n)$, but recycles the derivative $f'(x_n)$ that was already computed in the first step. This strategy is designed to increase the order of convergence while preserving the computational efficiency index.

However, utilizing the tangent slope evaluated at the previous point $x_n$ to approximate the behavior at the new point $z_n$ introduces an approximation error. Therefore, in the third step, we introduce a linear weight function $G(\eta_n) = 1 + \alpha\eta_n$. This function serves as a correction factor to compensate for the approximation error caused by the use of the frozen derivative $f'(x_n)$.

The selection of a linear polynomial with a free parameter $\alpha$ is sufficient to satisfy the asymptotic condition and it also provides the algebraic flexibility to eliminate the principal error term.

Consequently, the complete proposed three-step iterative method is defined as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - H(\eta_n)\frac{f(y_n)}{f'(x_n)}$$

$$x_{n+1} = z_n - \left(1 + \alpha\eta_n\right)\frac{f(z_n)}{f'(x_n)}$$

where the derivative ratio is defined as $\eta_n = \frac{f'(y_n)}{f'(x_n)} - 1$. The weight function $H(\eta_n)$ in the second step satisfies $H\left(0\right) = 1$, $H'\left(0\right) = -1$, and $H''\left(0\right) = \frac{5}{2}$, while $\alpha \in \mathbb{R}$ is a free parameter.

### 3.1. Convergence Analysis

To analyze the convergence order of the proposed three-step method, we present the following theorem. It proves that the method achieves sixth- and seventh-order convergence, depending on the chosen parameter $\alpha$.

**Theorem 2.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function on an open interval $I$. Furthermore, let $r \in I$ be a simple root of the nonlinear equation $f(x) = 0$. If the initial guess $x_0$ is sufficiently close to $r$ and the weight function $H(\eta_n)$ satisfies $H\left(0\right) = 1$, $H'\left(0\right) = -1$, and $H''\left(0\right) = 5/2$, then the proposed three-step method has:

- Sixth-order convergence for any parameter $\alpha \neq -1$.
- Seventh-order convergence specifically when $\alpha = -1$.

**Proof.** Let $e_n = x_n - r$ be the error at the $n$-th iteration. Since $r$ is a simple root of $f(x) = 0$, we have $f(r) = 0$ and $f'(r) \neq 0$. Then we define the normalized Taylor coefficient as

$$c_k = \frac{f^{(k)}(r)}{k!\, f'(r)}, \qquad k = 2,3, \ldots.$$

Moreover, for brevity, let $h_3 = H^{(3)}\left(0\right)$. Since $x_n = e_n - r$, the Taylor expansion of $f(x_n)$ around the root $r$ is given by:

$$f(x_n) = f'(r)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O\left(e_n^6\right)\right).$$

Furthermore,

$$f'(x_n) = f'(r)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O\left(e_n^5\right)\right).$$

By dividing these two expansions, we obtain:

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(-4c_2^3 + 7c_2c_3 - 3c_4\right)e_n^4 + \left(8c_2^4 - 20c_2^2c_3 + 10c_2c_4 + 6c_3^2 - 4c_5\right)e_n^5 + \mathcal{O}\left(e_n^6\right).$$

The first step of the proposed method is given by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Thus,

$$y_n - r = e_n - \frac{f(x_n)}{f'(x_n)}.$$

Consequently,

$$y_n - r = c_2 e_n^2 - 2\left(c_2^2 - c_3\right)e_n^3 + \left(4c_2^3 - 7c_2c_3 + 3c_4\right)e_n^4 + \left(-8c_2^4 + 20c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5\right)e_n^5 + \mathcal{O}\left(e_n^6\right).$$

Hence,

$$y_n - r = \mathcal{O}\left(e_n^2\right).$$

Next, the expansions of $f(y_n)$ and $f'(y_n)$ are given by:

$$\begin{aligned} f(y_n) &= f'(r)\left[(y_n - r) + c_2(y_n - r)^2 + c_3(y_n - r)^3 + \mathcal{O}\left((y_n - r)^4\right)\right] \\ &= f'(r)\left[(y_n - r) + c_2(y_n - r)^2 + c_3(y_n - r)^3 + \mathcal{O}\left(e_n^8\right)\right], \end{aligned}$$

and

$$\begin{aligned} f'(y_n) &= f'(r)\left[1 + 2c_2(y_n - r) + 3c_3(y_n - r)^2 + \mathcal{O}\left((y_n - r)^3\right)\right] \\ &= f'(r)\left[1 + 2c_2(y_n - r) + 3c_3(y_n - r)^2 + \mathcal{O}\left(e_n^6\right)\right]. \end{aligned}$$

Since $\eta_n = \frac{f'(y_n)}{f'(x_n)} - 1$, by substituting the expansions of $f'(y_n)$ and $f'(x_n)$ we obtain:

$$\eta_n = -2c_2 e_n + \left(6c_2^2 - 3c_3\right)e_n^2 + \left(-16c_2^3 + 16c_2c_3 - 4c_4\right)e_n^3 + \mathcal{O}\left(e_n^4\right).$$

Therefore, $\eta_n = \mathcal{O}\left(e_n^4\right)$.

Since the weight function satisfies

$$H(0) = 1, \qquad H'(0) = -1, \qquad H''(0) = -\frac{5}{2},$$

the Taylor expansion of $H(\eta_n)$ around $\eta_n = 0$ is given by:

$$H(\eta_n) = 1 - \eta_n + \frac{5}{4}\eta_n^2 + \frac{h_3}{6}\eta_n^3 + \mathcal{O}\left(\eta_n^4\right).$$

By substituting the expansion of $\eta_n$ we obtain:

$$H(\eta_n) = 1 + 2c_2e_n + \left(-c_2^2 + 3c_3\right)e_n^2 - \frac{4h_3c_2^3 + 42c_2^3 + 3c_2c_3 - 12c_4}{3}e_n^3 + \mathcal{O}\left(e_n^4\right).$$

Next, from the expansions of $f(y_n)$ and $f'(x_n)$, we get:

$$\frac{f(y_n)}{f'(x_n)} = c_2e_n^2 + \left(-4c_2^2 + 2c_3\right)e_n^3 + \left(13c_2^3 - 14c_2c_3 + 3c_4\right)e_n^4 + \left(-38c_2^4 + 64c_2^2c_3 - 20c_2c_4 - 12c_3^2 + 4c_5\right)e_n^5 + \mathcal{O}\left(e_n^6\right).$$

The second step of the proposed method is given by:

$$z_n = y_n - H(\eta_n)\frac{f(y_n)}{f'(x_n)}.$$

Thus,

$$z_n - r = (y_n - r) - H(\eta_n)\frac{f(y_n)}{f'(x_n)}.$$

By substituting the expansions of $y_n - r$, $H(\eta_n)$, and $\frac{f(y_n)}{f'(x_n)}$, we obtain:

$$z_n - r = \frac{c_2^2}{3}\left(4h_3c_2^2 + 42c_2^2 - 3c_3\right)e_n^5 + \mathcal{O}\left(e_n^6\right).$$

Hence,

$$z_n - r = \mathcal{O}\left(e_n^5\right).$$

Since $z_n - r = \mathcal{O}\left(e_n^5\right)$, the following holds for the analysis up to order $e_n^7$:

$$\frac{f(z_n)}{f'(x_n)} = (z_n - r)\left[1 - 2c_2e_n + \left(4c_2^2 - 3c_3\right)e_n^2 + \mathcal{O}\left(e_n^3\right)\right].$$

The correction factor in the third step is $1 + \alpha\eta_n$.

Since $\eta_n = -2c_2e_n + \left(6c_2^2 - 3c_3\right)e_n^2 + \mathcal{O}\left(e_n^3\right)$, we have:

$$1 + \alpha\eta_n = 1 - 2c_2e_n + \alpha\left(6c_2^2 - 3c_3\right)e_n^2 + \mathcal{O}\left(e_n^3\right).$$

Consequently,

$$(1 + \alpha\eta_n)\frac{f(z_n)}{f'(x_n)} = (z_n - r)\left[1 - 2(\alpha + 1)c_2e_n + \left((4 + 10\alpha)c_2^2 - 3(\alpha + 1)c_3\right)e_n^2 + \mathcal{O}\left(e_n^3\right)\right].$$

The third step of the proposed method is given by:

$$x_{n+1} = z_n - (1 + \alpha\eta_n)\frac{f(z_n)}{f'(x_n)}.$$

Therefore,

$$e_{n+1} = x_{n+1} - r = z_n - r - (1 + \alpha\eta_n)\frac{f(z_n)}{f'(x_n)}.$$

By substituting the previous expansion we obtain:

$$e_{n+1} = (z_n - r) - (z_n - r)\left[1 - 2(\alpha+1)c_2e_n + \left((4+10\alpha)c_2^2 - 3(\alpha+1)c_3\right)e_n^2 + \mathcal{O}\left(e_n^3\right)\right].$$

Thus,

$$e_{n+1} = (z_n - r)\left[2(\alpha+1)c_2e_n - \left((4+10\alpha)c_2^2 - 3(\alpha+1)c_3\right)e_n^2 + \mathcal{O}\left(e_n^3\right)\right].$$

Since $z_n - r = \frac{c_2^3}{3}\left(4h_3c_2^2 + 42c_2^2 - 3c_3\right)e_n^5 + \mathcal{O}\left(e_n^6\right)$,

for $\alpha \neq -1$ we obtain:

$$e_{n+1} = \frac{2(\alpha+1)c_2^3}{3}\left(4h_3c_2^2 + 42c_2^2 - 3c_3\right)e_n^6 + \mathcal{O}\left(e_n^7\right).$$

Since $h_3 = H^{(3)}(0)$, we have:

$$e_{n+1} = \frac{2(\alpha+1)c_2^3}{3}\left(4H^{(3)}(0)c_2^2 + 42c_2^2 - 3c_3\right)e_n^6 + \mathcal{O}\left(e_n^7\right).$$

Therefore, for $\alpha \neq -1$ the method has a sixth-order convergence.

Next, for $\alpha = -1$, the sixth-order term vanishes because it has the factor $\alpha + 1$. From the previous error expansion, we have:

$$e_{n+1} = (z_n - r)\left[2(\alpha+1)c_2e_n - \left((4+10\alpha)c_2^2 - 3(\alpha+1)c_3\right)e_n^2 + \mathcal{O}\left(e_n^3\right)\right].$$

If $\alpha = -1$, then

$$2(\alpha+1)c_2e_n = 0,$$

and

$$(4+10\alpha)c_2^2 - 3(\alpha+1)c_3 = (4-10)c_2^2 - 0 = -6c_2^2.$$

Consequently,

$$e_{n+1} = (z_n - r)\left[-\left(-6c_2^2\right)e_n^2 + \mathcal{O}\left(e_n^3\right)\right],$$

or

$$e_{n+1} = (z_n - r)\left[6c_2^2 e_n^2 + \mathcal{O}\left(e_n^3\right)\right].$$

Since we obtained

$$z_n - r = \frac{c_2^3}{3}\left(4h_3 c_2^2 + 42c_2^2 - 3c_3\right)e_n^5 + \mathcal{O}\left(e_n^6\right),$$

by substituting it to the expansion of $e_{n+1}$ we get:

$$e_{n+1} = \left[\frac{c_2^3}{3}\left(4h_3 c_2^2 + 42c_2^2 - 3c_3\right)e_n^5 + \mathcal{O}\left(e_n^6\right)\right]\left[6c_2^2 e_n^2 + \mathcal{O}\left(e_n^3\right)\right].$$

The principal term of this multiplication is

$$e_{n+1} = 6c_2^2 \cdot \frac{c_2^2}{3}\left(4h_3 c_2^2 + 42c_2^2 - 3c_3\right)e_n^7 + \mathcal{O}\left(e_n^8\right).$$

Thus,

$$e_{n+1} = 2c_2^4\left(4h_3 c_2^2 + 42c_2^2 - 3c_3\right)e_n^7 + \mathcal{O}\left(e_n^8\right).$$

Since $h_3 = H^{(3)}(0)$, we have:

$$e_{n+1} = 2c_2^4\left(4H^{(3)}(0)c_2^2 + 42c_2^2 - 3c_3\right)e_n^7 + \mathcal{O}\left(e_n^8\right).$$

Therefore, for $\alpha = -1$ the method has a seventh-order convergence.

### 3.2. Numerical Simulations

To validate the theoretical results and evaluate the performance of the proposed iterative scheme, numerical simulations were conducted to compare it with the existing fifth-order methods developed by Liu et al., a classical seventh-order method by Bi, Ren, and Wu, and a recently published seventh-order method by Tao and Wan.

The comparative methods from Liu are denoted as LM1 for

$$H(\eta_n) = 1 - \eta_n + \frac{5}{4}\eta_n^2 + \frac{1}{6}\eta_n^3$$

and LM2 for

$$H(\eta_n) = \frac{1 + (\alpha - 1)\eta + \left(\beta - \alpha - \frac{5}{4}\right)\eta^2}{1 + \alpha\eta + \beta\eta^2}$$

where $\alpha = \beta = 1$. Furthermore, the seventh-order method proposed by Bi, Ren, and Wu is denoted as BRWM, and the seventh-order method proposed by Tao and Wan where $\beta = 0$ is denoted as TWM.

For the proposed method, we denote PM1 for $\alpha = 1$ and PM2 for the special case of $\alpha = -1$. The simulations were performed on the following nonlinear test functions:

$$f_1(x) = x^3 + 4x^2 - 10,$$

$$f_2(x) = x^2 - e^x - 3x + 2,$$

$$f_3(x) = (x-1)^3 - 2,$$

$$f_4(x) = (x+2)e^x - 1,$$

$$f_5(x) = \sin^2(x) - x^2 + 1,$$

where the corresponding solutions are $r_1 = 1.3652300134$, $r_2 = 0.2575302854$, $r_3 = 2.2599210499$, $r_4 = -0.442854401002$, and $r_5 = 1.4044916482$. To obtain the numerical solutions, the iterations were terminated if one of the following stopping criteria is satisfied:

1. The number of iterations reaches a maximum of 100,
2. $|f(x_n)| < 10^{-200}$,
3. $|x_n - x_{n-1}| < 10^{-200}$.

The computational results are shown in Table 1, where for each method, the table displays the number of iterations ($n$) required to satisfy the stopping criteria, the absolute value of function $|f(x_n)|$, the distance between two consecutive approximations $|x_n - x_{n-1}|$, the Approximated Computational Order of Convergence (ACOC), and the Number of Function Evaluations (NFE).

**Table 1.** Comparison of Numerical Results Among Various Iterative Methods.

| $f_i(x)$ | $x_0$ | Methods | $n$ | $\|f(x_n)\|$ | $\|x_n - x_{n-1}\|$ | ACOC | NFE |
|---|---|---|---|---|---|---|---|
| $f_1(x)$ | $-0.3$ | LM1 | 15 | $5.5372e-219$ | $1.3094e-44$ | 5.00 | 60 |
| | | LM2 | 9 | $2.5517e-279$ | $1.0701e-70$ | 4.00 | 36 |
| | | BRWM | 31 | $2.1224e-329$ | $1.5798e-47$ | 6.93 | 124 |
| | | TWM | 11 | $2.0706e-518$ | $1.2355e-74$ | 6.99 | 44 |
| | | PM1 | 39 | $1.334e-798$ | $6.1069e-134$ | 6.00 | 195 |
| | | PM2 | 73 | $8.1338e-434$ | $8.8644e-63$ | 6.98 | 365 |
| $f_2(x)$ | 0.0 | LM1 | 4 | $1.1511e-825$ | $3.3992e-165$ | 5.00 | 16 |
| | | LM2 | 4 | $3.2284e-316$ | $3.1986e-79$ | 4.00 | 16 |
| | | BRWM | 3 | $4.1709e-448$ | $4.1963e-64$ | 7.00 | 12 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | TWM | 3 | $8.9491e-489$ | $7.8385e-70$ | 6.98 | 12 |
| | | PM1 | 3 | $2.353e-272$ | $1.7553e-45$ | 6.06 | 15 |
| | | PM2 | 3 | $5.2469e-440$ | $6.4516e-63$ | 7.05 | 15 |
| $f_3(x)$ | 3.0 | LM1 | 5 | $9.38e-712$ | $3.1911e-143$ | 5.00 | 20 |
| | | LM2 | 6 | $1.0323e-744$ | $4.5631e-187$ | 4.00 | 24 |
| | | BRWM | 4 | $4.7569e-1001$ | $6.606e-158$ | 7.00 | 16 |
| | | TWM | 4 | $2.3198e-878$ | $3.7767e-126$ | 7.00 | 16 |
| | | PM1 | 4 | $1.7889e-261$ | $1.6716e-44$ | 5.98 | 20 |
| | | PM2 | 4 | $2.639e-515$ | $1.6016e-74$ | 6.99 | 20 |
| $f_4(x)$ | 3.5 | LM1 | 7 | $3.6166e-386$ | $5.7696e-78$ | 5.00 | 28 |
| | | LM2 | 9 | $4.2154e-451$ | $1.6619e-113$ | 4.00 | 36 |
| | | BRWM | 7 | $4.6935e-581$ | $1.7645e-83$ | 7.00 | 28 |
| | | TWM | 5 | $9.0224e-399$ | $1.7618e-57$ | 6.98 | 20 |
| | | PM1 | 7 | $9.2215e-949$ | $6.3316e-159$ | 6.00 | 35 |
| | | PM2 | 6 | $7.842e-472$ | $3.3999e-68$ | 6.99 | 30 |
| $f_5(x)$ | 1.0 | LM1 | 6 | $3.0974e-674$ | $1.1685e-135$ | 5.00 | 24 |
| | | LM2 | 6 | $7.3292e-260$ | $8.8508e-66$ | 4.00 | 24 |
| | | BRWM | 4 | $2.3785e-1001$ | $8.9325e-153$ | 7.00 | 16 |
| | | TWM | 4 | $7.1523e-638$ | $9.0784e-92$ | 7.00 | 16 |
| | | PM1 | 5 | $2.8865e-348$ | $6.4352e-59$ | 5.99 | 25 |
| | | PM2 | 5 | $2.3785e-1001$ | $2.1661e-185$ | 7.00 | 25 |

Based on Table 1, the proposed methods PM1 and PM2 demonstrate stable convergence across all tested functions. Although BRWM and TWM occasionally require fewer iterations due to their theoretical design that utilizing four function evaluations per iteration compared to PM2, which requires five evaluations to reach the same order. However, it is observed that for certain scenarios, such as evaluating $f_1(x)$ with $x_0 = -0.3$, the proposed methods need much higher number of iterations. This behavior naturally occurs when the initial guess is located far enough from the exact root.

To demonstrate the pure asymptotic convergence of the proposed scheme, an additional experiment on $f_1(x)$ using a closer initial guess $x_0 = 1.0$ is presented in Table 2.

**Table 2.** Convergence Behaviour of the Iterative Methods for $f_1(x)$ with $x_0 = 1.0$

| Methods | $n$ | $\|f(x_n)\|$ | $\|x_n - x_{n-1}\|$ | ACOC | NFE |
|---|---|---|---|---|---|
| LM1 | 5 | $4.029e-885$ | $7.7526e-178$ | 5.00 | 20 |
| LM2 | 5 | $1.226e-319$ | $8.9095e-81$ | 4.00 | 20 |
| BRWM | 3 | $3.3469e-254$ | $8.7326e-37$ | 7.12 | 12 |
| TWM | 3 | $1.0394e-207$ | $3.0036e-30$ | 7.17 | 12 |
| PM1 | 4 | $1.7835e-284$ | $2.9751e-48$ | 5.98 | 20 |
| PM2 | 4 | $8.2097e-621$ | $1.7137e-89$ | 7.00 | 20 |

As shown in Table 2, when the initial guess is sufficiently close to the exact root, the required iterations for PM1 and PM2 drop to only 4 iterations. Instead of requiring a high number of iterations as seen in the distant initial guess scenario, PM1 and PM2 can now quickly achieve extreme precision, reducing the absolute residual $|f(x_n)|$ to $1.7835 \cdot 10^{-284}$ and $8.2097 \cdot 10^{-621}$, respectively, in just four steps. Furthermore, these numerical simulations also validate Theorem 2, where the Approximated Computational Order of Convergence (ACOC) firmly confirms that PM1 has a sixth-order convergence and PM2 has a seventh-order convergence.

## 4. CONCLUSIONS

This paper successfully developed a new family of three-step iterative methods for solving nonlinear equations by modifying an existing fifth-order method through the introduction of a third step. By utilizing a frozen derivative concept combined with a linear weight function $G(\eta_n) = 1 + \alpha\eta_n$ as a correction factor, the method efficiently increase the convergence rate without any additional derivative evaluations. Theoretical convergence analysis proves that the proposed method has a sixth-order convergence for any parameter $\alpha \neq -1$ and accelerates to seventh-order convergence specifically for $\alpha = -1$.

To validate the theoretical analysis, numerical simulations were conducted. The performance of the proposed methods (PM1 and PM2) wa compared against existing methods including Liu's fifth-order methods (LM1 and LM2), Bi-Ren-Wu's classical seventh-order method (BRWM), and Tao-Wan's recently developed seventh-order method (TWM). The computational results show that the proposed iterative scheme is competitive, demonstrates rapid convergence and produces highly accurate approximations with a minimal number of iterations. Furthermore, the proposed method preserves the theoretical Approximated Computational Order of Convergence (ACOC). Thus, the proposed method serves as an efficient, stable, and powerful alternative for solving nonlinear equations.

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