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Modifications of Distances in Metric Spaces with Invisible Holes

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ABSTRACT

Invisible holes in metric spaces are defined. Holes are classified as hi holes and non hi holes in metric spaces. Modified metrics are provided to measure distance around holes, but inside metric spaces. Some properties of modified metrics are derived.

***Keywords*:** Metric spaces, Convex sets, Completion, Integrals.

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1. INTRODUCTION

There are many old branches of geometry, like differential geometry and analytical geometry. There is an old book published in 1953 with the heading "Distance Geometry" for geometry in metric spaces. So, there is another branch of geometry which discusses concepts in metric spaces [4, 6]. One can easily define geometrical concepts of spheres and balls in metric spaces. There are defined concepts like convex sets, curvature, normal structure, gaps, porosity, mid points and so on in metric spaces. The most recent article [9] uses the word "holes' in its title, but uses the word "gaps" in the text, because research articles mostly use the word "gaps," which represent numbers to measure sizes of undefined "holes." It has been mentioned in the article [9] that properties of metric spaces have been studied in metric spaces only in terms of the points of those metric spaces or only in terms of the subsets of those metric spaces, by pointing out that gaps of metric spaces in earlier articles were studied in larger metric spaces containing those basic metric spaces. For every metric space, a larger metric space that contains the basic metric space is the completion of the basic metric space. The completion of a metric space is constructed by means of equivalence classes containing Cauchy sequences of the basic metric space. So, if a Cauchy sequence in a metric space does not converge in the metric space, then the Cauchy sequence may be realized or treated as a point hole in the metric space [5]. The aim of the present article is to consider how to realize all holes in a metric space and how to modify the given metric in the metric space so that the distance is covered only through the metric space without passing over/through invisible holes.

For example, when a bus travels from a point x to another point y, the distance to be travelled should be mentioned only by means of the distance along the road through which the bus travels, not by means of the shortest distance straight line. Similarly, there is a realization that the distance should be measured through a shortest path inside a metric space without passing over holes, as given in Figure 1. So, there is a need to modify metrics [7]. One may observe from Figure 1 that there is no need to change the distance when the hole contains only one point. So, there is no need to give importance to one point holes. But, points in completion have to be considered in terms of Cauchy sequences to identify all holes in a metric space.

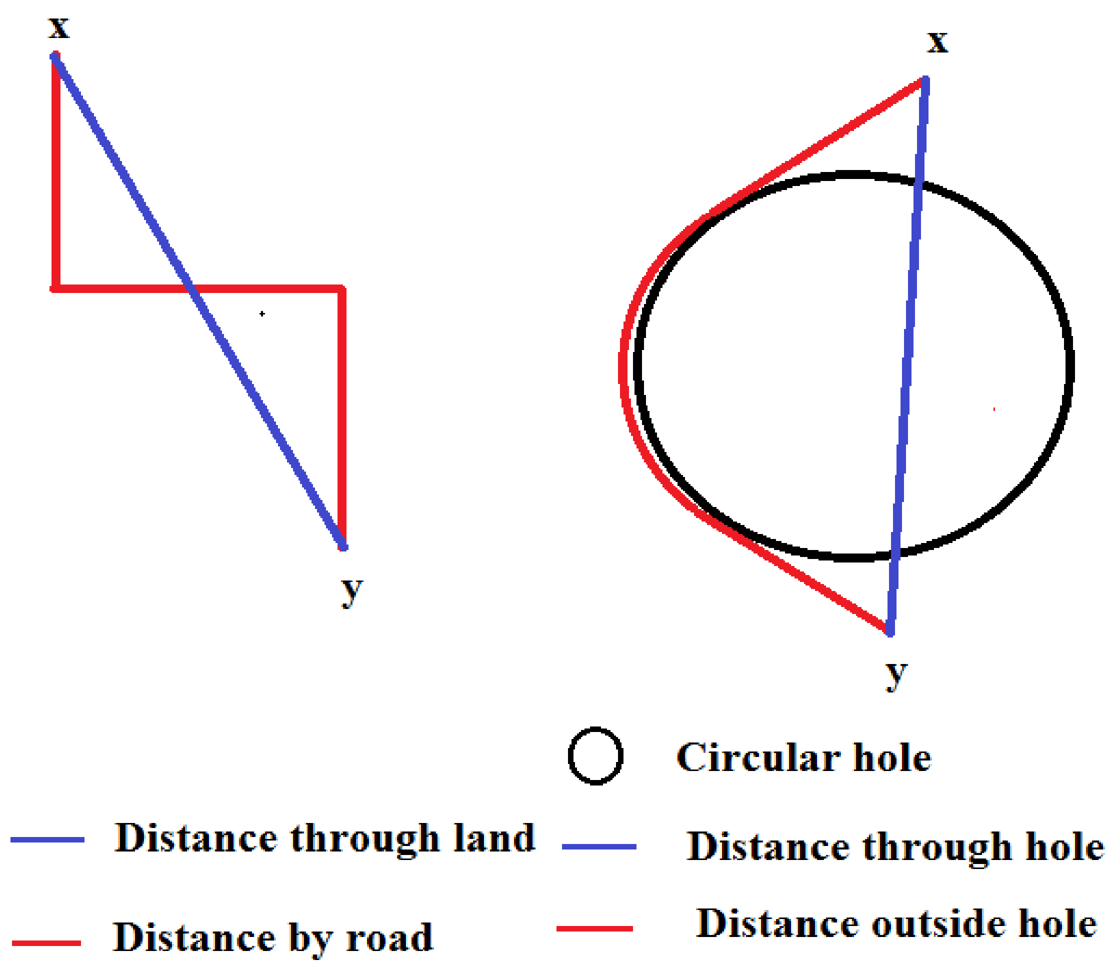

Figure 1. Permissible routes.

A fixed point theorem was proved in the article [13] with the keywords convex sets and normal structure in normed spaces. To generalize this theorem to metric spaces, researchers had to introduce the concepts of convex sets and normal structure in metric spaces [14]. For example, the convex hull of a subset A of a metric space was considered as the intersection of all closed balls containing the subset A, with an understanding that closed balls are considered as closed convex subsets of the metric space [1, 18]. Apart from generalizing results of differential geometry to distance geometry, researchers extended geometrical concepts to metric spaces to obtain fixed point results. The purpose of the present article is just to propose a definition for distance that is suitable for all metric spaces, as that one is explained in Figure 1. More specifically, metrics those are suitable for metric spaces having holes. Our big problem is to explain a procedure to understand invisible holes in metric spaces. Single point holes can be explained as Cauchy sequences. Our problem is to explain all holes, which are unions of single point holes. So, the next Section 2 defines invisible holes in metric spaces. Section 3 defines convex sets and convex hulls in metric spaces. Section 4 provides modifications required in metrics. Section 5 explains that the definition providing modifications in metrics is related to integrals.

2. HOLES

Let us define "holes" in metric spaces gradually. "Holes" are invisible parts of a metric space (X, d). So, members of holes cannot be members of (X, d). However, holes of (X, d) should be realized and identified only by the members of (X, d). Let us define holes gradually under these restrictions.

Definition 1: Let (X, d) be a metric space. Suppose that there are two distinct points $x_1, x_2 \in X$ such that there is no $x_3 \in X$ such that $0 < d(x_1, x_3) < d(x_1, x_2)$ and such that $0 < d(x_2, x_3) < d(x_1, x_2)$. Then it is said that there is a strong hole in X associated with the pair (x_1, x_2), and it is said that this pair is a boundary pair of a strong hole in X.

Definition 2: Let (X, d) be a metric space. Suppose that there are two distinct points $x_1, x_2 \in X$ such that there is no $x_3 \in X$ such that $0 < d(x_1, x_3) < d(x_1, x_2)$, $0 < d(x_2, x_3) < d(x_1, x_2)$, and such that $d(x_1, x_2) = d(x_1, x_3) + d(x_2, x_3)$. Then, it is said that there is a hole in X associated with the pair (x_1, x_2), and it is said that this pair is a boundary pair of a hole in X.
See Figure 2 for an understanding of these concepts given in Definition 1 and Definition 2. In Figure 2, outside of the holes is a metric space. To understand holes, there is a need to understand boundaries of holes.

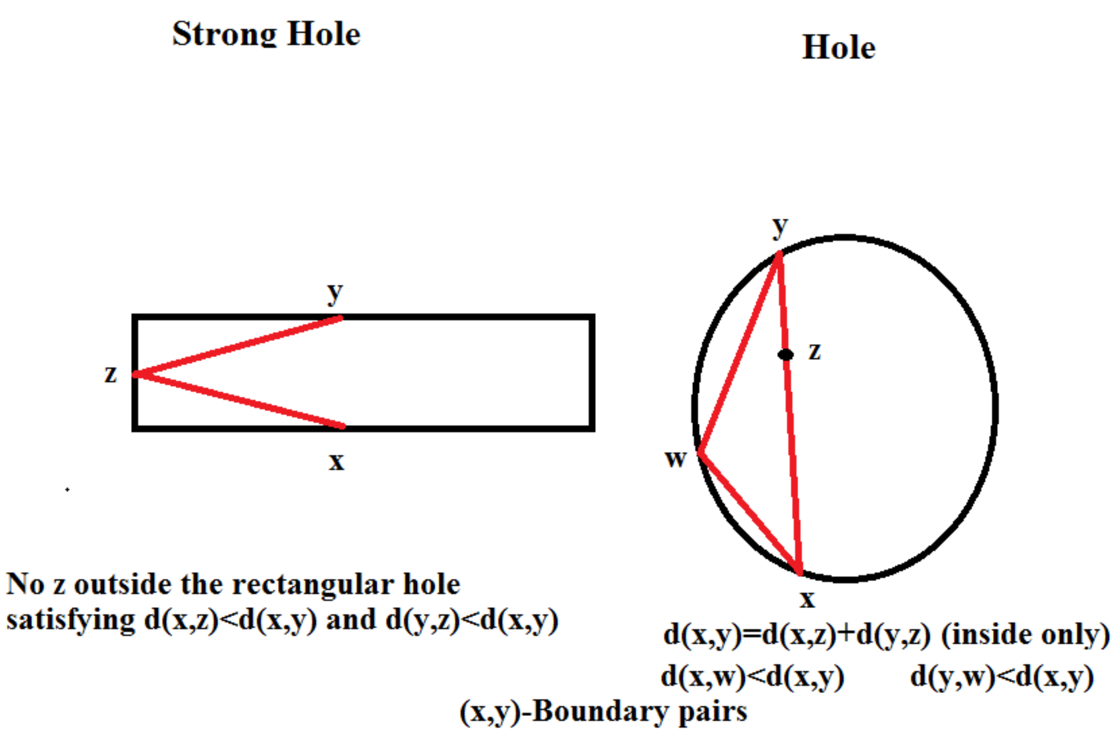

Figure 2. Holes and Strong Holes.

Let (X, d) be a metric space. Let us consider a boundary pair (x_1, x_2) of a strong hole. Let $\mathcal{C} = \{ F \subseteq X : x_1, x_2 \in F$, and there are finitely many points $y_1, y_2, \ldots, y_n$ in X such that $(x_1, y_1), (y_1, y_2), (y_2, y_3), \ldots . (y_{n-1}, y_n), (y_n, x_2)$ are boundary pairs of strong holes in $X\}$. Let $B = \bigcup_{F \in \mathcal{C}} F$. Then B may be considered as a part of boundary of the strong hole associated with the pair (x_1, x_2).

Let (X, d) be a metric space. Let us consider a boundary pair (x_1, x_2) of a hole. Let $\mathcal{C} = \{ F \subseteq X: x_1, x_2 \in F$, and there are finitely many points $y_1, y_2, \dots, y_n$ in X such that $(x_1, y_1), (y_1, y_2), (y_2, y_3), \dots . (y_{n-1}, y_n), (y_n, x_2)$ are boundary pairs of holes in $X\}$. Let $B = \bigcup_{F \in \mathcal{C}} F$. Then B may be considered as a part of boundary of the hole associated with the pair (x_1, x_2).

For complete boundaries, and for all holes, there is a need to go for completion of metric spaces. All holes and all parts of boundaries of all holes can be identified easily in completions, and thereby the restrictions of them to basic metric spaces provide almost all holes and all parts of boundaries in basic metric spaces. Let us first recall the standard construction of the completion of a given metric space [16, 23].

Let (X, d) be a metric space. Let $\mathcal{C}$ be the collection of all Cauchy sequences in (X, d). Define an equivalence relation $\sim$ in $\mathcal{C}$ by writing $(x_i)_{i=1}^{\infty} \sim (y_i)_{i=1}^{\infty}$ when and only when $d(x_i, y_i) \to 0$ as $i \to \infty$. Let us use the notation $[(x_i)_{i=1}^{\infty}]$ for the equivalence class containing $(x_i)_{i=1}^{\infty}$. Let $\tilde{X}$ denote the collection of all these equivalence classes. A metric $d: \tilde{X} \times \tilde{X} \to [0, \infty)$ is defined by $d([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}]) = \lim_{i \to \infty} d(x_i, y_i)$. Let us use the same notation d for metrics in both X and $\tilde{X}$. Let us define a mapping $T: X \to \tilde{X}$ by $T(x) = [(x_i)_{i=1}^{\infty}]$ with $x_i = x, \forall i$ and for all $x \in X$. Then $T: X \to \tilde{X}$ is an isometry and $T(X)$ is dense in $(\tilde{X}, d)$. So, X is identified with $T(X)$ so that (X, d) is identified as a dense subspace of its completion $(\tilde{X}, d)$. More specifically, every element $x \in X$ can be identified with any Cauchy sequence that is equivalent to the constant sequence with elements x. Let us use these notations regarding completions of metric spaces. Let us observe that if (Tx_1, Tx_2) is a boundary pair of a hole in completion of (X, d), then (x_1, x_2) is a boundary pair of a hole in (X, d).

Let (X, d) be a metric space and let $(\tilde{X}, d)$ be its completion. Let us consider a boundary pair $([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}])$ of a strong hole in $(\tilde{X}, d)$. Let $\mathcal{C} = \{ F \subseteq \tilde{X}: [(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}] \in F$, and there are finitely many points $z_1, z_2, \dots, z_n$ in $\tilde{X}$ such that $([(x_i)_{i=1}^{\infty}], z_1), (z_1, z_2), (z_2, z_3), \dots . (z_{n-1}, z_n), (z_n, [(y_i)_{i=1}^{\infty}])$ are boundary pairs of strong holes in $\tilde{X}\}$. Let $B = \bigcup_{F \in \mathcal{C}} F$. Then B may be considered as a part of boundary of the strong hole associated with the pair $([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}])$ in $(\tilde{X}, d)$.

The complete definition now follows.

Definition 3: Let (X, d) be a metric space and let $(\tilde{X}, d)$ be its completion. Let us consider a boundary pair $([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}])$ of a hole in $(\tilde{X}, d)$. Let $\mathcal{C} = \{ F \subseteq \tilde{X}: [(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}] \in F$, and there are finitely many points $z_1, z_2, \dots, z_n$ in $\tilde{X}$ such that $([(x_i)_{i=1}^{\infty}], z_1), (z_1, z_2), (z_2, z_3), \dots . (z_{n-1}, z_n), (z_n, [(y_i)_{i=1}^{\infty}])$ are boundary pairs of holes in $\tilde{X}\}$. Let $B = \bigcup_{F \in \mathcal{C}} F$. Then B is defined as the boundary of the hole associated with the pair $([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}])$ in $(\tilde{X}, d)$. A boundary pair $([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}])$ in $(\tilde{X}, d)$ defines a boundary B, and this boundary is now defined as a hi HOLE in $(\tilde{X}, d)$.

Roughly speaking, boundaries are used to identify missing holes. Let us first observe that any two hi HOLES in $(\tilde{X}, d)$ are either identical or disjoint, and so Definition 3 is meaningful. The word "hi" is used for an abbreviation of "having interior." To define holes in (X, d), let us also use the notation $d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})$ for the value $d([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}])$.

Let (X, d) be a metric space. Suppose that there are two non equivalent Cauchy sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in X such that there is no Cauchy sequence $(z_i)_{i=1}^{\infty}$ in X such that $0 < d((x_i)_{i=1}^{\infty}, (z_i)_{i=1}^{\infty}) < d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})$ and such that $0 < d((y_i)_{i=1}^{\infty}, (z_i)_{i=1}^{\infty}) < d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})$. Then it said that there is a strong hole in X associated with the pair $((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})$, and it is said that this pair is a boundary pair of a strong hole in X.

Definition 4: Let (X, d) be a metric space. Suppose that there are two non equivalent Cauchy sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in X such that there is no Cauchy sequence $(z_i)_{i=1}^{\infty}$ in X such that $0 < d((x_i)_{i=1}^{\infty}, (z_i)_{i=1}^{\infty}) < d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})$, $0 < d((y_i)_{i=1}^{\infty}, (z_i)_{i=1}^{\infty}) < d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})$, and such that $d((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}) = d((x_i)_{i=1}^{\infty}, (z_i)_{i=1}^{\infty}) + d((y_i)_{i=1}^{\infty}, (z_i)_{i=1}^{\infty})$. Then it said that there is a hole in X associated with the pair $((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty})$, and it is said that this pair is a boundary pair of a hole in X. Let us consider the boundary B associated with the pair $([(x_i)_{i=1}^{\infty}], [(y_i)_{i=1}^{\infty}])$ given in Definition 3. This boundary B is defined as the boundary of a hi hole in (X, d), as well as the hi hole of (X, d). This is done because equivalence classes of Cauchy sequences are defined only in terms of members of (X, d).
With such a same satisfaction, let us now define non hi holes in a metric space (X, d).

Definition 5: Let (X, d) be a metric space. Let C denote the collection of all equivalence classes $[(x_i)_{i=1}^{\infty}]$ of Cauchy sequences in (X, d) such that $[(x_i)_{i=1}^{\infty}]$ are not any member of any hi hole in (X, d), and such that $(x_i)_{i=1}^{\infty}$ do not converge in (X, d). Let us consider C as a subset of $(\tilde{X}, d)$ and let us find all maximal connected components of C relative to $(\tilde{X}, d)$. All these components are called non hi holes of the metric space (X, d).

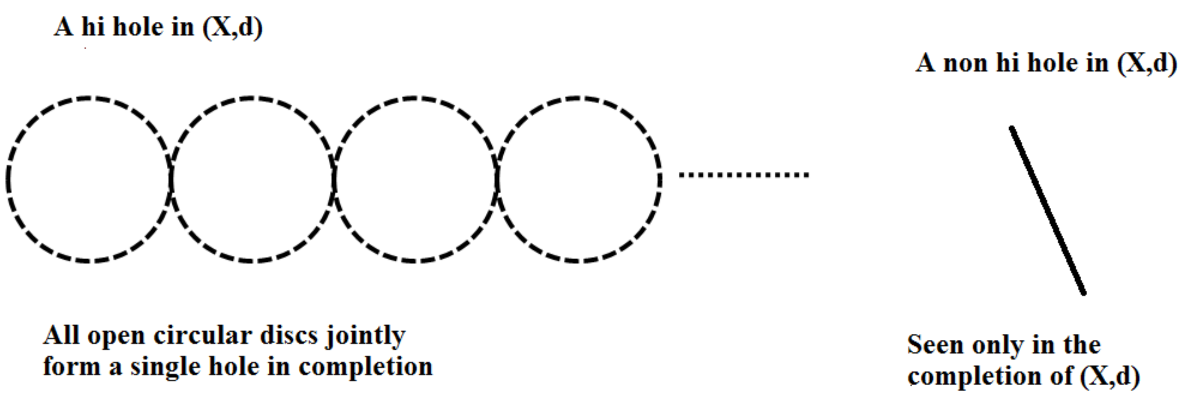

Figure 3. Hi holes and non hi holes.

All holes of a metric space (X, d) have been defined. These holes are of two types, namely, hi holes and non hi holes. These two types of holes are illustrated in Figure 3. All infinitely many disjoint open discs given in the first part of Figure 3 should be treated as a single hi hole, because there is a common boundary between any two successive disjoint open discs. A single straight line segment given in the second part of Figure 3 cannot be a subset of (X, d), but it can be a subset of $(\tilde{X}, d)$. In this connection, the following remark should be recorded in view of Definition 5.

Remark 1: There are no non hi holes in complete metric spaces.

3. CONVEX SETS

When hi holes have been defined, a relation of the type "$d(x,y) = d(x,z) + d(z,y)$" has been associated. Geometrically this may be viewed in the real line like this: the point z lies in an interval with end points x and y. A subset E of a real vector space is called a convex set [15] when every point of the type $z = \lambda x + (1-\lambda)y$ lies in E, for every λ satisfying $0 \leq \lambda \leq 1$, whenever $x, y \in E$. In geometric words, it said that the line segment with end points x and y is contained in a convex set E, whenever $x, y \in E$. In a metric space (X,d), a line segment with end points x and y is the collection of all points z satisfying $d(x,y) = d(x,z) + d(z,y)$.

Let us use the use the following notation for any two points x and y in a metric space (X,d).

Closed line segment: $[x,y] = \{z \in X: d(x,y) = d(x,z) + d(z,y)\}$.

A closed line segment should be a closed set in a metric space. For, let us consider a sequence $(z_i)_{i=1}^{\infty}$ in $[x,y]$ of a metric space (X,d). Suppose that this sequence converges to a point z in (X,d). Then $d(x,z_i) \to d(x,z)$ and $d(z_i,y) \to d(z,y)$ as $i \to \infty$. Then $d(x,z_i) + d(z_i,y) \to d(x,z) + d(z,y)$ as $i \to \infty$. However, $d(x,y) = d(x,z_i) + d(z_i,y), \forall i$. Therefore, $d(x,y) = d(x,z) + d(z,y)$. This proves that $z \in [x,y]$ so that $[x,y]$ is a closed subset of (X,d). Observe that $x, y \in [x,y]$.

Let (X,d) be a metric space. Let E be a subset of X. Then E is said to be a convex set, if $[x,y] \subseteq E$, whenever $x, y \in E$. Intersection of any collection of convex sets is a convex set in a metric space. Let us call the smallest convex set [3, 12] containing a given subset F of a metric space (X,d) as the convex hull of F in (X,d). It is possible to Construct the convex hull of a given set in two ways, internally and externally.

Let F be a subset of a metric space (X,d) and let E be the convex hull of F in (X,d). Then E is the intersection of all convex sets containing F. This is an external construction. Let $F_1 = \cup_{x,y \in F}[x,y]$, $F_{i+1} = \cup_{x,y \in F_i}[x,y]$, $\forall i = 1, 2, 3, ...$ Then $E = \cup_{i=1}^{\infty} F_i$. This is an internal construction.

4. INDUCED CURVED METRICS

It has been mentioned in Section 1 that there is a need to find metrics without passing over holes with reference to Figure 1. It is being difficult to find such metrics for all metric spaces. However, it is possible to find such metrics in Euclidean spaces with holes. Let us begin with known concepts.

Definition 6: Let (X,d) be a metric space. Let x_1 and x_2 be distinct points of X. Let $\varepsilon > 0$ be given. A finite sequence $x_1, y_1, y_2, \ldots, y_n, x_2$ of distinct points in X is called an ε chain of the pair (x_1, x_2) when $d(y_i, y_{i+1}) \leq \varepsilon, \forall i = 0, 1, 2, \ldots, n$, with $y_0 = x_1$ and $y_{n+1} = x_2$. The length of this chain is $\sum_{i=0}^{n} d(y_i, y_{i+1})$. The greatest lower bound value of all lengths of such ε chains of (x_1, x_2) is denoted by $d_\varepsilon(x_1, x_2)$. That is, $d_\varepsilon(x_1, x_2) = \text{Inf}\,\{\sum_{i=0}^{n} d(y_i, y_{i+1})$: all ε chains $x_1, y_1, y_2, \ldots, y_n, x_2$ of $(x_1, x_2)\}$.

A pair (x_1, x_2) of distinct points of (X,d) is said to be completely chainable if this pair has at least one ε chain in X, for every $\varepsilon > 0$. For a completely chainable pair (x_1, x_2), let us define $D(x_1, x_2) = Sup_{\varepsilon > 0} d_\varepsilon(x_1, x_2)$.

For a given $\varepsilon > 0$, a metric space (X,d) is said to be ε chainable if every pair (x_1, x_2) of distinct elements of X has an ε chain in X. A metric space is said to be completely chainable if every pair of distinct elements of the metric space is completely chainable in that metric space.

Proposition 1: Let (X, d) be an ε chainable metric space. Then (X, d_ε) is a metric space.

Proof: Let $x, y, z \in X$. Fix $\delta > 0$. Then there are finite sequences $x, x_1, x_2, \dots, x_m, y$ and $y, y_1, y_2, \dots, y_n, z$ in X such that $d(y_i, y_{i+1}) \le \varepsilon$, $\forall i = 0, 1, 2, \dots, n$, with $y_0 = y$ and $y_{n+1} = z$, $d(x_i, x_{i+1}) \le \varepsilon$, $\forall i = 0, 1, 2, \dots, m$, with $x_0 = x$ and $x_{m+1} = y$, $\sum_{i=0}^{n} d(y_i, y_{i+1}) \le d_\varepsilon(y, z) + \delta$, and such that $\sum_{i=0}^{m} d(x_i, x_{i+1}) \le d_\varepsilon(x, y) + \delta$. Then $\sum_{i=0}^{m} d(x_i, x_{i+1}) + \sum_{i=0}^{n} d(y_i, y_{i+1}) \le d_\varepsilon(x, y) + d_\varepsilon(y, z) + 2\delta$, when $x_0 = x$, $x_{m+1} = y_0$ and $y_{n+1} = z$. Therefore, $d_\varepsilon(x, z) \le d_\varepsilon(x, y) + d_\varepsilon(y, z) + 2\delta$, for every $\delta > 0$. This proves that $d_\varepsilon(x, z) \le d_\varepsilon(x, y) + d_\varepsilon(y, z)$.

Note that $d(x, y) \le d_\varepsilon(x, y)$ in an ε chainable metric space.

Proposition 2: Let (X, d) be a completely chainable metric space. Then $D: X \times X \to [0, \infty]$ is an extended metric in the sense that (i) $D(x, y) = 0$ if and only if $x = y$ in X; (ii) $D(x, y) = D(y, x)$, $\forall x, y \in X$, and (iii) $D(x, y) \le D(x, z) + D(z, y)$, $\forall x, y, z \in X$. But $D(x, y)$ may take the value $+\infty$ for some $x, y \in X$.

Proof: Let $x, y, z \in X$. Then, $d_\varepsilon(x, z) \le d_\varepsilon(x, y) + d_\varepsilon(y, z)$, $d_\varepsilon(x, y) \le D(x, y)$, and $d_\varepsilon(y, z) \le D(y, z)$, for every $\varepsilon > 0$. Thus, $d_\varepsilon(x, z) \le D(x, y) + D(y, z)$, for every $\varepsilon > 0$. This proves that $D(x, y) \le D(x, z) + D(z, y)$.

Note that $d(x, y) \le d_\varepsilon(x, y) \le D(x, y)$ in a completely chainable metric space.

The possibility of assuming the value $D(x, y) = \infty$ cannot be ruled out as it is seen in the following Example 1 with reference to Figure 4.

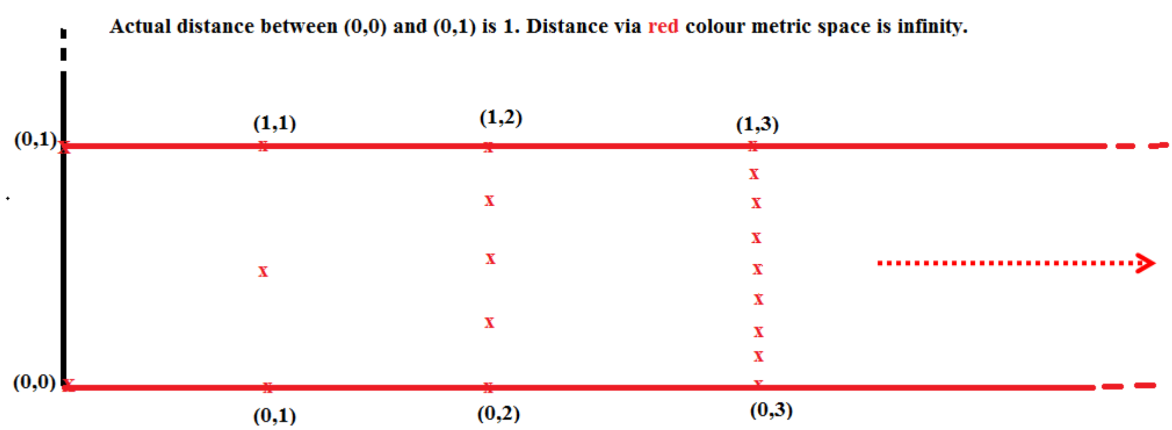

Figure 4. Possible infinity value for D in a subspace of the plane.

Example 1: Let d now denote the usual Euclidean metric in the usual Euclidean plane. Let $X = \{(x, 0), (x, 1) : 0 \le x < \infty\} \cup \left\{\left(1, \frac{1}{2}\right), \left(2, \frac{1}{4}\right), \left(2, \frac{2}{4}\right), \left(2, \frac{3}{4}\right), \left(3, \frac{1}{8}\right), \left(3, \frac{2}{8}\right), \dots, \left(3, \frac{7}{8}\right), \dots .\right\}$. Then $d_\varepsilon\big((0,0), (0,1)\big)$ has the following values. It is 1, when $\varepsilon = 1$. It is 3, when $\varepsilon = \frac{1}{2}$. It is 5, when $\varepsilon = \frac{1}{4}$. It is 7, when $\varepsilon = \frac{1}{8}$. The values go on in this pattern. So, $D\big((0,0), (0,1)\big) = +\infty$.

Definition 7: Let (X, d) be a metric space. Let x_1 and x_2 be distinct points of X. Let $\varepsilon > 0$ be given. A finite sequence $x_1, y_1, y_2, \dots, y_n, x_2$ of distinct points in X is called a convex ε chain of the pair (x_1, x_2) when $d(y_i, y_{i+1}) \le \varepsilon$, $\forall i = 0, 1, 2, \dots, n$, with $y_0 = x_1$ and $y_{n+1} = x_2$, and when $d(x_1, x_2) = \sum_{i=0}^{n} d(y_i, y_{i+1})$.
A pair (x_1, x_2) of distinct points of (X, d) is said to be completely convex chainable if this pair has at least one convex ε chain in X, for every $\varepsilon > 0$.

For a given $\varepsilon > 0$, a metric space (X, d) is said to be convex ε chainable if every pair (x_1, x_2) of distinct elements of X has a convex ε chain in X. A metric space is said to be completely convex chainable if the metric space is convex ε chainable for every $\varepsilon > 0$.

Proposition 3: Let (X, d) be a convex ε chainable metric space. Then $d = d_\varepsilon$.
Proof: It follows from Definition 6 and Definition 7.

Proposition 4: Let (X, d) be a completely convex chainable metric space. Then $d = D$.
Proof: It follows from Definition 6 and Definition 7.

Remark 2: Any completely convex chainable metric space has no hi holes. This follows from Definition 4 and Definition 7.

Proposition 5: Let (X, d) be a complete metric space with no holes. Then, it is a completely convex chainable metric space so that $d = D$.

Proof: The proof is based on a transfinite induction process applied successively inside another transfinite induction process. Let us fix a pair (x_1, x_2) of distinct points of (X, d) and an $\varepsilon > 0$ such that $d(x_1, x_2) > \varepsilon$. Since there are no hi holes, there is a sequence $x_3, x_4, x_5, \ldots$ of distinct points of the closed line segment $[x_1, x_2]$ in (X, d) such that $0 < d(x_1, x_{i+1}) < d(x_1, x_i)$, and such that $d(x_1, x_i) = d(x_1, x_{i+1}) + d(x_i, x_{i+1})$, for every $i = 2, 3, 4, \ldots$. Let the decreasing sequence $\left(d(x_1, x_i)\right)_{i=2}^{\infty}$ converge to a so that $\sum_{i=2}^{\infty} d(x_i, x_{i+1})$ converges to $d(x_1, x_2) - a$. If $a = 0$, then there is a point y_1, in the closed line segment $[x_1, x_2]$ such that $0 < d(x_1, y_1) \leq \varepsilon$. Suppose $a > 0$. Then the sequence $(x_i)_{i=2}^{\infty}$ converges to some point x_2^* in the closed line segment $[x_1, x_2]$ such that $d(x_1, x_2^*) = a$. If $a \leq \varepsilon$, then again there is a point $y_1 = x_2^*$ in the closed line segment $[x_1, x_2]$ such that $0 < d(x_1, y_1) \leq \varepsilon$. Let us suppose that $a > \varepsilon$. Let us apply the previous process for the closed line segment $[x_1, y_1]$ and let us find a point y_2 in the line segment $[x_1, y_1]$ such that $0 < d(x_1, y_2) < d(x_1, y_1)$ with two possibilities: (i) $d(x_1, y_2) \leq \varepsilon$; or (ii) $d(x_1, y_2) > \varepsilon$. Let us consider the second possibility $d(x_1, y_2) > \varepsilon$, and let us apply the process for the closed line line segment $[x_1, y_2]$ and let us find a point y_3 in the line segment $[x_1, y_2]$ such that $0 < d(x_1, y_3) < d(x_1, y_2)$ with two possibilities: (i) $d(x_1, y_3) \leq \varepsilon$; or (ii) $d(x_1, y_3) > \varepsilon$. Let us continue this process infinitely. Let the decreasing sequence $\left(d(x_1, y_i)\right)_{i=2}^{\infty}$ converge to b so that$d(x_2, y_1) + \sum_{i=1}^{\infty} d(y_i, y_{i+1})$ converges to $d(x_1, x_2) - b$. If $b = 0$, then there is a point z_1, in the closed line segment $[x_1, x_2]$ such that $0 < d(x_1, z_1) \leq \varepsilon$. Suppose $b > 0$. Then the sequence $(y_i)_{i=2}^{\infty}$ converges to some point x_3^* in the closed line segment $[x_1, x_2]$ such that $d(x_1, x_3^*) = b$. If $b \leq \varepsilon$, then again there is a point $z_1 = x_3^*$ in the closed line segment $[x_1, x_2]$ such that $0 < d(x_1, z_1) \leq \varepsilon$. Let us suppose that $b > \varepsilon$. Let us continue this infinite process infinitely again and again. But this process gets terminated at a countable step in the transfinite process, because the set of all rational numbers is countable. So, one can have the following conclusion.

There is a point u_1 in the line segment $[x_1, x_2]$ such that $0 < d(x_1, u_1) \leq \varepsilon$. If $d(x_1, x_2) - d(x_1, u_1) \leq \varepsilon$, then there is a convex ε chain x_1, u_1, x_2 for the interval $[x_1, x_2]$. Otherwise, there is a point u_2 in line segment $[u_1, x_2]$ such that $0 < d(u_1, u_2) \leq \varepsilon$. If $d(x_1, x_2) - d(x_1, u_1) - d(u_1, u_2) \leq \varepsilon$, then there is a convex ε chain x_1, u_1, u_2, x_2 for the interval $[x_1, x_2]$. A transfinite induction process can be applied with this suggestion and the previous suggestion to handle limiting cases to find a convex ε chain $x_1, v_1, v_2, \ldots, v_m, x_2$ for the interval $[x_1, x_2]$ and in the interval $[x_1, x_2]$.

Remark 3: Let $(\tilde{X}, d)$ be the completion of a given metric space (X, d). Then there are d_ε and D on $\tilde{X}$, corresponding to d on $\tilde{X}$. Then their restrictions coincide with the corresponding d_ε and D derived in (X, d). This one follows from the fact that (X, d) is dense in its completion and from the definitions for d and D. Suppose $S: (X, d) \to (Y, d_Y)$ be an onto mapping such that $d_Y(Sx, Sy) \leq d(x, y)$, $\forall x, y \in X$. If $d_{Y\varepsilon}$ and D_Y are extended metrics on Y, corresponding to d and D on X, then $d_{Y\varepsilon}(Sx, Sy) \leq d_\varepsilon(x, y)$, $\forall x, y \in X$, and $D_Y(Sx, Sy) \leq D(x, y)$, $\forall x, y \in X$.
Let us explore Example 1 in a different direction.

Proposition 6: Let (X, d) be a metric space. Let A and B be disjoint non empty subsets of (X, d) whose union is X. Suppose that (A, d) and (B, d) are completely chainable metric spaces. Suppose that $\inf\{d(x, y): x \in A, y \in B\} = 0$. Suppose that there is no bounded sequence $(x_i)_{i=1}^{\infty}$ in A and there is no bounded sequence $(y_i)_{i=1}^{\infty}$ in B such that $d(x_i, y_i) \to 0$ as $i \to \infty$. Then, $D(x, y) = \infty$, $\forall x \in A$ and $\forall y \in B$.

Proof: Since $\inf\{d(x, y): x \in A, y \in B\} = 0$, then for any $x \in A$, and for any $y \in B$, the pair (x, y) is completely chainable, so that $D(x, y)$ is defined. Suppose that $D(x, y) = M < \infty$ for some $x \in A$ and for some $y \in B$. Then $0 < d_\varepsilon(x, y) \leq M$, $\forall \varepsilon > 0$. For every i, let $\varepsilon_i = \frac{1}{i}$. For every i, there is an ε_i chain $x, u_1, u_2, \dots, u_{n_i}, y$ for the pair (x, y) with $u_0 = x$ and $u_{n_{i+1}} = y$ such that $\sum_{j=0}^{n_i} d(u_j, u_{j+1}) < d_{\varepsilon_i}(x, y) + \frac{1}{2}$ $< M + 1$. Let $x_i \in A$ and $y_i \in B$ be two successive members in that chain so that $d(x_i, y_i) \leq \varepsilon_i$. Then, $d(x_i, y_i) \to 0$ as $i \to \infty$. So, both sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ are unbounded. So, $\left(d(x, x_i)\right)_{i=1}^{\infty}$ and $\left(d(y, y_i)\right)_{i=1}^{\infty}$ are unbounded sequences, because $d(x, y)$ is finite. However, by the triangle inequality, $0 \leq d(x, x_i) \leq \sum_{j=0}^{n_i} d(u_j, u_{j+1}) < M + 1 < \infty$, $\forall i$. This is a contradiction, which proves the result.

Example 2: Let $A = \{(x, y): x \geq 0,\ y \geq e^{-x}\}$ and let $B = \{(x, y): x \geq 0,\ y \leq -e^{-x}\}$. Let $X = A \cup B$, and let d be the Euclidean metric on X. Then, $\inf\{d(x, y): x \in A, y \in B\} = 0$, $A \cap B = \emptyset$, and there is no bounded sequence $(x_i)_{i=1}^{\infty}$ in A and there is no bounded sequence $(y_i)_{i=1}^{\infty}$ in B such that $d(x_i, y_i) \to 0$ as $i \to \infty$. The boundary of only one hi hole in (X, d) is $\{(x, y): x \geq 0,\ y = e^{-x}\} \cup \{(x, y): x \geq 0,\ y = -e^{-x}\}$.

Proposition 7: Let (X, d) be a metric space. Let A and B be disjoint non empty subsets of (X, d) whose union is X. Suppose that (A, d) and (B, d) are completely chainable metric spaces. Suppose that $\inf\{d(x, y): x \in A, y \in B\} = 0$. Suppose that for every bounded subset E of (A, d) or (B, d), the set $\{D(x, y): x, y \in E\}$ is bounded. Suppose that there is a bounded sequence $(x_i)_{i=1}^{\infty}$ in (A, d) and there is a bounded sequence $(y_i)_{i=1}^{\infty}$ in (B, d) such that $d(x_i, y_i) \to 0$ as $i \to \infty$, $\sum_{i=1}^{\infty} d(x_i, x_{i+1}) < \infty$, and such that $\sum_{i=1}^{\infty} d(y_i, y_{i+1}) < \infty$. Then, $D(x, y) < \infty$, $\forall x \in A$ and $\forall y \in B$.

Proof: Fix $x \in A$ and $y \in B$. Then $E = \{x, x_1, x_2, \dots\}$ and $F = \{y, y_1, y_2, \dots\}$ are bounded sets. Then $\{D(u, v): u, v \in E\}$ and $\{D(u, v): u, v \in F\}$ are bounded. Let M be a finite positive constant such that $Sup\,\{D(u, v): u, v \in E\} \leq M$, and $Sup\,\{D(u, v): u, v \in F\} \leq M$. For a given $1 > \varepsilon > 0$, find an integer m such that $\sum_{i=m}^{\infty} d(x_i, x_{i+1}) < \frac{\varepsilon}{4}$ and such that $\sum_{i=m}^{\infty} d(y_i, y_{i+1}) < \frac{\varepsilon}{4}$. Find $n > m + 1$ such that $d(x_n, y_n) < \frac{\varepsilon}{4}$.

Then $d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) + d(x_n, y_n) + d(y_n, y_{n-1}) + \cdots + d(y_{m+2}, y_{m+1}) + d(y_{m+1}, y_m) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3}{4}\varepsilon$. Therefore, $d_\varepsilon(x_m, y_m) < \varepsilon < 1$. So, $d_\varepsilon(x, y) \leq d_\varepsilon(x, x_m) + d_\varepsilon(x_m, y_m) + d_\varepsilon(y_m, y) < 2M + 1$. Thus, $d_\varepsilon(x, y) < 2M + 1, \forall \varepsilon \in (0, 1)$, so that $D(x, y) \leq 2M + 1 < \infty$.

Example 3: Let $A = \{(x, y): x \geq 0,\ y \geq x, (x, y) \neq (0, 0)\}$ and let $B = \{(x, y): x \geq 0,\ y \leq -x\}$. Let $X = A \cup B$, and let d be the Euclidean metric on X. Then, $\inf\{d(x, y): x \in A, y \in B\} = 0$, $A \cap B = \emptyset$, and there is a bounded sequence $(x_i)_{i=1}^{\infty}$ in A and there is a bounded sequence $(y_i)_{i=1}^{\infty}$ in B such that $d(x_i, y_i) \to 0$ as $i \to \infty$, $\sum_{i=1}^{\infty} d(x_i, x_{i+1}) < \infty$, and such that $\sum_{i=1}^{\infty} d(y_i, y_{i+1}) < \infty$. Moreover, for every bounded subset E of (A, d) or (B, d), the set $\{D(x, y): x, y \in E\}$ is bounded. The boundary of only one hi hole in (X, d) is $\{(x, y): x \geq 0,\ y = x\} \cup \{(x, y): x \geq 0,\ y = -x\}$.

One can modify d as d_ε or D so that the objective expected in Section 1 related to Figure 1 is satisfied, by considering the results of the present section.

5. CURVED INTEGRALS

Definition 6 provides the definition of a particular case of curved integrals for the constant function 1 defined on a complete chainable metric space. This can be seen by comparing the following Definition 8 with Definition 6.

Definition 8: Let (X, d) be a completely chainable metric space. Let $f: X \to [0, \infty)$ be a given function. Let x_1 and x_2 be two given points in X. For a given $\varepsilon > 0$, let us consider an ε chain in the form $x_1, y_1, y_2, \ldots, y_n, x_2$ with $y_0 = x_1$ and $y_{n+1} = x_2$, for convenience. With this convention, let us define $i_\varepsilon(x_1, x_2)(f) = \text{Inf}\ \{\sum_{i=0}^{n} \big(f(y_i) \text{ or } f(y_{i+1})\big) d(y_i, y_{i+1})$: for every ε chain $x_1, y_1, y_2, \ldots, y_n, x_2$ of the pair $(x_1, x_2)\}$. Now, let us define the curved integral of the function f between x_1 and x_2 by $I(x_1, x_2)(f) = Sup_{\varepsilon > 0} i_\varepsilon(x_1, x_2)(f)$.

It follows from the proofs of Proposition 1 and Proposition 2 that $i_\varepsilon(x_1, x_2)(f) \leq i_\varepsilon(x_1, x_3)(f) + i_\varepsilon(x_3, x_2)(f)$, and $I(x_1, x_2)(f) \leq I(x_1, x_3)(f) + I(x_3, x_2)(f)$, for any three distinct points x_1, x_2, x_3.

It is possible to compare curved integrals with line integrals [8, 11, 17, 20]. However, they are compared with Riemann integrals, to avoid parametric representations of curves. Consider a Riemann integrable function $f: [a, b] \to [0, \infty)$. From the definition [2, 10, 19, 21, 22] of the Riemann integral for Riemann integrable functions, it follows that $\int_a^b f(t)dt = I(a, b)(f)$. For a given real valued function f, its positive part and negative part are defined by $f_+ = (|f| + f)/2$ and $f_- = (|f| - f)/2$. Then $f = f_+ - f_-$.

Definition 9: Let (X, d) be a completely chainable metric space. Let f be a given real valued function defined on X. Let x_1 and x_2 be two given points in X. Now, let us define the curved integral of the function f between x_1 and x_2 by $I(x_1, x_2)(f) = I(x_1, x_2)(f_+) - I(x_1, x_2)(f_-)$, provided the right hand side is meaningful.

6. CONCLUSION

Holes have been completely defined in metric spaces. One can consider the possibility of using d_ε or D for modifications of a given metric d, when the given metric space contains holes. One can consider $I(x_1, x_2)(f)$ as a possible generalization of line integrals.

Conflicts of Interests

The author declared no conflicts of interest.

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