Available online www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 213 (2026) 89-106 EISSN 2392-2192

Functions Well Defined Relative to a Function

Chinnadurai Ganesa Moorthy[1]

[1]Department of Mathematics, Alagappa University, Karaikudi – 630003, India

E-mail address: ganesamoorthyc@gmail.com

https://doi.org/10.65770/XKMP8410

ABSTRACT

Differentiation of a function f(x) with respect to a function g(x) is known as the ratio having numerator as the derivative of fwith respect to x and having denominator as the derivative of g with respect to x. This definition is modified as the limit of the ratio ((f(x)-f(s)/(g(x)-g(s)) when g(x) tends to g(s) in a meaningful way. Integration of a function with respect a function of bounded variation is known as a Riemann-Stieltjes integral. The same approach meant for Riemann-Stieltjes integrals is extended for more functions general than functions of bounded variation. These extensions are achieved through a classification of functions which are well defined with respect a given function. More precisely, a function f is said to be well defined with respect to g, when these functions have a common domain, if f(x)=f(y) whenever g(x)=g(y). Continuity of a function with respect to a function is also discussed. It is observed that this approach provides an approach that is easier than the existing ones. It is further observed that the definition of functions well defined with respect to a given function can be extended to a definition of functions well defined with respect to given two functions, and thereby the possibility for extending differentiation to partial differentiation and for extending integration to joint integration with respect to several given functions becomes positive.

Keywords**:** Well defined functions, Continuity, Differentiation, Integration.

(Received 10 January 2026; Accepted 16 February 2026; Date of Publication 14 March 2026)

1. INTRODUCTION

Ordinary usual differentiation is defined with respect a variable in the sense that derivative of a function of a single variable is obtained with respect to that variable. Usual Riemann integration for a function is also defined with respect to the variable for the function. This note is considered to discuss about differentiation and integration with respect to functions. This idea is a very old one. For example, $\frac{du}{dv}$ is considered as $\frac{\frac{du}{dx}}{\frac{dv}{dx}}$, when both u and v are functions of a common variable x. One may find such a derivative in derivations of prey-predator equations. In connection with integration, one may find a relation $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)dx$ in connection with Riemann-Stieltjes integrals. So, such things are commonly known. But this note is to review these ideas in a general way. For this purpose, functions f which are well defined with respect a given function g are considered, and then only for such functions $\frac{df}{dg}$ and $\int f\, dg$ are discussed. Even such functions can be converted into ordinary functions, and then known results can be applied to these ordinary functions to derive the results which are to be discussed in this note. But this note provides an exercise for a direct approach rather than going for converted functions. Training in direct approaches provides easy applicability of the theory.

2. FUNCTIONS WELL DEFINED RELATIVE TO A FUNCTION

Let us recall a definition for well defined functions. A function f is said to be well defined on its domain, when $f(x) = f(y)$ whenever $x = y$ in the domain.

Definition: Consider two functions $f: S \to X$ and $g: S \to Y$, having same domain. The function f is said to be well defined on S with respect to g (or relative to g), if $f(s_1) = f(s_2)$ whenever $g(s_1) = g(s_2)$ in Y with $s_1, s_2 \in S$. In this case, f is called a g-defined function.

Consider a g-defined function f given in the previous definition. Define a new function $h: g(S) \to X$ by $h(g(s)) = f(s), \forall s \in S$. This is a usual well defined function. Verification: If $g(s_1) = g(s_2)$, then $f(s_1) = f(s_2)$. Classical results can be applied to such functions h, and the results to be discussed in this note can be derived. However, training will take place only for direct approaches for easy applicability of the theory.

Let us consider one example: Consider one vector space N and two of its vector subspaces N_1 and N_2 such that $N_2 \subsetneq N_1 \subsetneq N$. Consider the natural quotient mappings $f: S \to X$ and $g: S \to Y$ with $S = N, X = N/N_1$ and $Y = N/N_2$. Then f is a g-defined mapping. Define $h: Y \to X$ by $h(x + N_2) = x + N_1$. Then h is a well defined mapping on $Y = g(S)$.

If $f(s) = \cos s$ and $g(s) = \sin s$, then $f(s)$ is not a g-defined function, when S is the entire real line. However, if $S = \left[0, \frac{\pi}{2}\right]$, then $f(s)$ is a g-defined function. This provides the importance of the specification of the sets S. Let us see the following sequence of mathematical expressions. $\frac{d(\cos s)}{d(\sin s)} = \frac{d(\sqrt{1-sin^2 s})}{d(sin\, s)} = \frac{1}{2\sqrt{1-sin^2 s}}(-2\, sin\, s)$.

To make these expressions as valid ones, it is known that the intervals in which s are varied should be analysed. Instead of this analysis, it is being simple to find the sets S on which $f(s) = \cos s$ is g-defined, when $g(s) = \sin s$, and the simplicity gets justification after introducing differentiation relative to a given function. Every differentiable function is a continuous function, when there are meaningful domain and co-domain. Let us first discuss functions continuous relative to a function.

3. CONTINUITY RELATIVE TO A FUNCTION

This section is based on topology and uniformity. One may refer to the book "General topology" of J.L. Kelly for these concepts.

Definition: Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Let $f: S \to (X, \tau_X)$ and $g: S \to (Y, \tau_Y)$ be given two functions on a set S. Suppose f is g-defined on S. Then the function f is said to be continuous at a point s_0 in S (or at $g(s_0)$), if for a given open neighbourhood U of $f(s_0)$ there is an open neighbourhood V of $g(s_0)$ such that $f\left(g^{-1}(V)\right) \subseteq U$. Also, f is said to be continuous on S (or on $g(S)$), relative to g, if for every $U \in \tau_X$, $g\left(f^{-1}(U)\right)$ is open in $g(S)$ with respect to the topology induced by τ_Y.

Remark: f is continuous on S relative to g if and only if f is continuous at every point of S, relative to g. Verification: Suppose f is continuous on S. Fix $s_0 \in S$, and an open neighbourhood U of $f(s_0)$. Let $W = g\left(f^{-1}(U)\right)$. Let $V \in \tau_Y$ be such that $V \cap g(S) = W$. Then $f\left(g^{-1}(V)\right) = f(g^{-1}(W)) = f\left(g^{-1}\left(g\left(f^{-1}(U)\right)\right)\right) = f\left(f^{-1}(U)\right) = U \cap f(S) \subseteq U$. This means that f is continuous at any given $s_0 \in S$. On the other hand, let us assume that f is continuous at every point of S, relative to g. Let $U \in \tau_X$. For each $s \in S$ with $f(s) \in U$, there is an open neighbourhood $V_{g(s)}$ of $g(s)$ such that $f\left(g^{-1}\left(V_{g(s)}\right)\right) \subseteq U$. Let $V = \bigcup_{f(s)\in U} V_{g(s)} \cap g(S)$. Then $V = g\left(f^{-1}(U)\right)$.

Remark: Let $\tau_S = \{g^{-1}(V): V \in \tau_Y\}$. Then f is continuous on$S$ relative to g if and only if $f: (S, \tau_S) \to (X, \tau_X)$ is continuous. If d is a metric on Y that defines the topology τ_Y, then define $\rho\left(g^{-1}\left(g(s_1)\right), g^{-1}\left(g(s_2)\right)\right) = d(y_1, y_2), with\ y_1 = g(s_1), y_2 = g(s_2)$, on the collection of all sets of the form $g^{-1}\left(g(s)\right)$. Then define $\sigma(t_1, t_2) = \rho\left(g^{-1}\left(g(s_1)\right), g^{-1}\left(g(s_2)\right)\right)$, when $t_1 \in g^{-1}\left(g(s_1)\right)$ and when $t_2 \in g^{-1}\left(g(s_2)\right)$. Then σ need not be a metric. However, define open balls with respect to σ, and then the collection of all possible union of such open balls forms the topology τ_S. Since σ is not a metric, in general, there is a need to go for topologies apart from metrics.

Proposition: Suppose $f: S \to (X, \tau_X)$ is a g-continuous mapping, when $g: S \to (Y, \tau_Y)$ is a given mapping. Let $h: (X, \tau_X) \to (Z, \tau_Z)$ be a continuous function with a topological space(Z, τ_Z) as co-domain. Then the composition mapping $(h \circ f): S \to (Z, \tau_Z)$ is a g-continuous function.

Proof: If $g(s_1) = g(s_2)$, then $f(s_1) = f(s_2)$ so that $h\left(f(s_1)\right) = h\left(f(s_2)\right)$. That is, $h \circ f$ is a g-mapping. Moreover $(h \circ f): (S, \tau_S) \to (Z, \tau_Z)$ is a continuous mapping. So, $(h \circ f): S \to (Z, \tau_Z)$ is a g-continuous function. It is also possible to derive the following result on continuity of "compositions".

Proposition:

(a) Suppose (S, τ_S) is a topological group under $+$, $s_0 \in S$ is a fixed point, and if $f: S \to (X, \tau_X)$ is continuous relative to $g: S \to (Y, \tau_Y)$. If Y is a group under $+$, and $g(s_0 + s) = y_0 + g(s), \forall s \in S$, for some $y_0 \in Y$, then $f_1: (S, \tau_S) \to (X, \tau_X)$ defined by $f_1(s) = f(s_0 + s), \forall s \in S$, is g-continuous. If Y is a group under $+$, and $g(s_0 - s) = y_0 - g(s), \forall s \in S$, for some $y_0 \in Y$, then $f_2: (S, \tau_S) \to (X, \tau_X)$ defined by $f_2(s) = f(s_0 - s), \forall s \in S$, is g-continuous.

(*b*) Suppose (S, τ_S) is a topological vector space and c is a fixed scalar. Suppose $f: S \to (X, \tau_X)$ is continuous relative to $g: S \to (Y, \tau_Y)$. If Y is a vector space and $g(s_0 + c\, s) = x_0 + c\, g(s)$, then $f_3: (S, \tau_S) \to (X, \tau_X)$ defined by $f_3(s) = f(s_0 + c\, s), \forall s \in S$ is continuous relative to g.

(*c*) Suppose $f: S \to (X, \tau_X)$ is continuous relative to $g: S \to (Y, \tau_Y)$. Suppose in addition that (S, τ_S) is a topological algebra and y_0 is a fixed element in Y. If Y is an algebra and $g(s_0\, s) = y_0\, g(s), \forall s \in S$, then $f_4: (S, \tau_S) \to (X, \tau_X)$ defined by $f_4(s) = f(s_0\, s), \forall s \in S$, is continuous relative to g.

It is known that image of a connected (compact) topological space is a connected (compact) set under a continuous mapping. This result can be modified in the following form.

Proposition: Suppose $f: S \to (X, \tau_X)$ be continuous on S relative to $g: S \to (Y, \tau_Y)$. If $g(S)$ is a connected (compact) set with respect to the subspace topology induced by τ_Y, then $(f(S), \tau_X)$ is connected (compact).

This result will be used later along with the fact that the collection of all connected subsets in the real line with the usual topology are the collection of all intervals, and with the fact that the closed and bounded subsets of the real line are compact with respect to the usual topology. Let us now introduce uniform continuity relative to a function.

Definition: Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces with uniformities $\mathcal{U}$ and $\mathcal{V}$, respectively. Let τ_X and τ_Y be the topologies on X and Y induced by the uniformities $\mathcal{U}$ and $\mathcal{V}$, respectively. Let $f: S \to X$ and $g: S \to Y$ be given two functions on a set S and suppose as usual that f is g-defined. Let $\mathcal{W}$ be the uniformity on the set S generated by the family $\left\{\{(s_1, s_2): (g(s_1), g(s_2)) \in V\}: V \in \mathcal{V}\right\}$. Then $f: S \to (X, \mathcal{U})$ is said to be uniformly continuous relative to $g: S \to (Y, \mathcal{V})$, if $f: (S, \mathcal{W}) \to (X, \mathcal{U})$ is uniformly continuous. Equivalently, if for given $U \in \mathcal{U}$, there is a $V \in \mathcal{V}$ such that $\{(f(s_1), f(s_2)): (g(s_1), g(s_2)) \in V\} \subseteq U$.

Let τ_X, and τ_Y be the topologies on X, and Y induced by the uniformities $\mathcal{U}$, and $\mathcal{V}$, respectively. Then it can be verified that τ_S is the topology induced by the map $g: S \to (Y, \tau_Y)$ on S, then this topology coincides with the topology induced by the uniformity $\mathcal{W}$, that has been mentioned in the previous definition. Moreover, if $f: S \to (X, \mathcal{U})$ is uniformly continuous relative to $g: S \to (Y, \mathcal{V})$, then $f: S \to (X, \tau_X)$ is continuous relative to $g: S \to (Y, \tau_Y)$. This remark is quite essential in proving the next result with these notations.

Proposition: If $f: S \to (X, \tau_X)$ is continuous relative to $g: S \to (Y, \tau_Y)$, and if $g(S)$ is compact with respect to the subspace topology induced by τ_Y, then $f: S \to (X, \mathcal{U})$ is uniformly continuous relative to $g: S \to (Y, \mathcal{V})$.

This result will be used later for the real line with the usual uniformity and with the usual topology. The uniform convergence in the next proposition is either on S, or on $g(S)$, and moreover, "sequence" may also be replaced by "net".

Proposition: Let $f_n: S \to (X, \tau_X), n = 1, 2, 3, \ldots$ be a sequence of functions which are continuous at a point s_0, relative to $g: S \to (Y, \tau_Y)$. Suppose that this sequence converges uniformly to a function $f: S \to (X, \tau_X)$. Then $f: S \to (X, \tau_X)$ is also continuous at s_0, relative to $g: S \to (Y, \tau_Y)$.

Corollary: Let $f_n: S \to \mathbb{R}, n = 1, 2, 3, \ldots$, be a sequence of functions which are continuous at a point s_0, relative to $g: S \to (Y, \tau_Y)$, when $\mathbb{R}$ is endowed with the usual topology. Suppose that the sequence $\sum_{i=1}^{n} f_i$, $n = 1, 2, 3, \ldots$, converges uniformly to a function $f: S \to \mathbb{R}$. Then $f: S \to \mathbb{R}$ is also continuous at s_0, relative to $g: S \to (Y, \tau_Y)$.

4. DIFFERENTIATION WITH RESPECT TO A FUNCTION

Definition: Let $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ be two functions on a fixed set S, when the real line $\mathbb{R}$ is considered with the usual topology. Suppose that f is a g-defined function. Fix $s_0 \in S$. Suppose that every neighbourhood of $g(s_0)$ contains infinitely many points $g(s)$ in the range $g(S)$. Then f is said to be differentiable at $g(s_0)$ (or at s_0) relative to g (or with respect to g), if there is a real number A such that

$$\lim_{\substack{g(s) \to g(s_0) \\ g(s) \neq g(s_0)}} \frac{f(s) - f(s_0)}{g(s) - g(s_0)} = A$$

in the sense that for a given $\varepsilon > 0$, there is a $\delta > 0$ such that $\left|\frac{f(s)-f(s_0)}{g(s)-g(s_0)} - A\right| < \varepsilon$, whenever $0 < |g(s) - g(s_0)| < \delta$ in $g(S)$. In this case, let us denote A as $f^{(g)}(s_0)$ and call it derivative of f at s_0 relative to g (or with respect to g) (or, g-derivative of f at s_0).

Note that, if $g(s_1) = g(s_0)$, then $f(s_1) = f(s_0)$ so that $\frac{f(s)-f(s_1)}{g(s)-g(s_1)} = \frac{f(s)-f(s_0)}{g(s)-g(s_0)}$, when the denominators are not equal to 0. Thus, the phrase "differentiable at s_0" can also be used without ambiguity instead of the phrase "differentiable at $g(s_0)$" in the previous definition. Similarly, it is also possible to replace $f^{(g)}(s_0)$ by $f^{(g)}\big(g(s_0)\big)$. If $f^{(g)}(s)$ exists for every $s \in S$, then $f^{(g)}: S \to \mathbb{R}$ is also ag-defined function. The phrase " $\left|\frac{f(s)-f(s_0)}{g(s)-g(s_0)} - A\right| < \varepsilon$, whenever $0 < |g(s) - g(s_0)| < \delta$ in $g(S)$" in the previous definition implies the following inequality: $|f(s) - f(s_0)| \leq (\varepsilon + A)|g(s) - g(s_0)|$, whenever $|g(s) - g(s_0)| < \delta$ in $g(S)$. This proves that f is continuous at s_0, relative to g, whenever f is differentiable at s_0, relative to g. Let us also observe that the condition that "every neighbourhood of $g(s_0)$ contains infinitely many points $g(s)$ in the range $g(S)$" is automatically satisfied when $g(S)$ is an interval. Moreover, if $g(s_0)$ is an interior point in an interval contained in $g(S)$, then every neighbourhood of $g(s_0)$ contains infinitely many points of $g(S)$ in the left side of $g(s_0)$ and infinitely many points of $g(S)$ in the right side of $g(s_0)$. This fact will be used to establish that the relative derivative is zero at a point where a relative maximum (or minimum) is reached. It is also possible to extend the definition to the functions $g: S \to (Y, \tau_Y)$, when (Y, τ_Y) is a topological vector space over the real field instead of the functions $g: S \to \mathbb{R}$. It is also possible to extend the definition to the functions $g: S \to \mathbb{C}$, when the complex field $\mathbb{C}$ is endowed with the usual topology, instead of the functions $g: S \to \mathbb{R}$. But these two possible extensions will not be considered in this note. Let us discuss the following example with this assurance.

Example: Let $S=\mathbb{R}$. Define $g: S \rightarrow \mathbb{R}$ by $g(s)=|s|, \forall s \in S$. Consider $f_1: S \rightarrow \mathbb{R}$ given by $f_1(s)=s^2, \forall s \in S$. If $g(s_1)=g(s_2)$, then $f_1(s_1)=f_1(s_2)$ so that f_1 is a g-defined function. Now, $\frac{f_1(s)-f_1(0)}{g(s)-g(0)}=\frac{s^2-0^2}{|s|-|0|}=\frac{|s|^2}{|s|}=|s|$, for $s \neq 0$. Thus, if $g(s) \rightarrow g(0)$ (with $s \neq 0$), or if $|s| \rightarrow 0$ with respect to the usual topology of the real line, then $f_1^{(g)}(0)$ exists and it is equal to 0 of the real line. If $s_0 \neq 0$, and if $|s_0| \neq |s| \neq 0$, then $\frac{f_1(s)-f_1(s_0)}{g(s)-g(s_0)}=\frac{s^2-s_0^2}{|s|-|s_0|}=\frac{|s|^2-|s_0|^2}{|s|-|s_0|}=|s|+|s_0|$. Thus, if $g(s) \rightarrow g(s_0)$ or if $|s| \rightarrow |s_0|$, then $f_1^{(g)}(s_0)$ exists and it is equal to $2|s_0|$, when $s_0 \neq 0$. In general, $f_1^{(g)}(s)=2|s|, \forall s \in S=\mathbb{R}$. Consider $f_2: S \rightarrow \mathbb{R}$ given by $f_2(s)=\sqrt{|s|}$, the positive square root of $|s|$, for every $s \in S$. Note that f_2 is also a g-defined function. Now, $\frac{f_2(s)-f_2(0)}{g(s)-g(0)}=\frac{\sqrt{|s|}-\sqrt{|0|}}{|s|-|0|}=\frac{1}{\sqrt{|s|}}$, for $s \neq 0$. When $g(s) \rightarrow g(0)$ (with $s \neq 0$), $\frac{f_2(s)-f_2(0)}{g(s)-g(0)} \rightarrow +\infty$ so that $f_2^{(g)}(0)$ does not exist. If $s_0 \neq 0$, and if $|s_0| \neq |s| \neq 0$, then $\frac{f_2(s)-f_2(s_0)}{g(s)-g(s_0)}=\frac{\sqrt{|s|}-\sqrt{|s_0|}}{|s|-|s_0|}=\frac{\sqrt{|s|}-\sqrt{|s_0|}}{\left(\sqrt{|s|}\right)^2-\left(\sqrt{|s_0|}\right)^2}=\frac{1}{\sqrt{|s|}+\sqrt{|s_0|}}$. Thus, for $s \neq 0, f_2^{(g)}(s)=\frac{1}{2\sqrt{|s|}}$ exists. Note that $f_1(s)=(g(s))^2$ and $f_2(s)=\sqrt{g(s)}$, for every $s \in S$. Also, $f_1^{(g)}(s)=2g(s), \forall s \in S$ and $f_2^{(g)}(s)=\frac{1}{2\sqrt{g(s)}}$, for every $s \neq 0$ in S.

It is possible to prove the following relations by using the arguments used in usual classical real analysis. Proofs are omitted in view of the modified arguments used in the previous example.

Proposition: Let $S=\mathbb{R}$. Let $g: S \rightarrow \mathbb{R}$ be any given function. Suppose that every neighbourhood of every point in $g(S)$ contains infinitely many points of $g(S)$.

(a) For a given real number α, if $f: S \rightarrow \mathbb{R}$ is defined by $f(s)=(g(s))^\alpha, \forall s \in S$, then $f^{(g)}(s)=\alpha(g(s))^{\alpha-1}$, provided $(g(s))^\alpha$ and $(g(s))^{\alpha-1}$are meaningful.
(b) If $f: S \rightarrow \mathbb{R}$ is defined by $f(s)=\sin(g(s)), \forall s \in S$, then $f^{(g)}(s)=\cos(g(s)), \forall s \in S$.
(c) If $f: S \rightarrow \mathbb{R}$ is defined by $f(s)=\exp(g(s)), \forall s \in S$, then $f^{(g)}(s)=\exp(g(s)), \forall s \in S$.
(d) In general, if for given real scalars $a_0, a_1, a_2, a_3, \ldots$ such that $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}=0$, a function $f: S \rightarrow \mathbb{R}$ is defined by $f(s)=\sum_{n=0}^{\infty} a_n(g(s))^n, \forall s \in S$, then $f^{(g)}(s)=\sum_{n=1}^{\infty} n a_n(g(s))^{n-1}, \forall s \in S$.
However, the following two propositions and some more results may be required for the proof of the previous proposition.

Proposition: Let $g: S \rightarrow \mathbb{R}$ be any given function. Suppose $f_1: S \rightarrow \mathbb{R}$ and $f_2: S \rightarrow \mathbb{R}$ are differentiable at $s_0 \in S$, relative to the function $g: S \rightarrow \mathbb{R}$. Then the followings are true.
(a) $(f_1+f_2): S \rightarrow \mathbb{R}$ is differentiable at s_0, and $(f_1+f_2)^{(g)}(s_0)=f_1^{(g)}(s_0)+f_2^{(g)}(s_0)$.
(b) $(cf_1): S \rightarrow \mathbb{R}$ is differentiable at s_0, and $(cf_1)^{(g)}(s_0)=cf_1^{(g)}(s_0)$, for any real number c.
(c) $(f_1 f_2): S \rightarrow \mathbb{R}$ is differentiable at s_0, and $(f_1 f_2)^{(g)}(s_0)=f_1^{(g)}(s_0) f_2(s_0)+f_1(s_0) f_2^{(g)}(s_0)$.
(d) Suppose $f_2(s_0) \neq 0$. Then $(f_1/f_2): S \rightarrow \mathbb{R}$ is differentiable at s_0, and $(f_1/f_2)^{(g)}(s_0)=\frac{f_1^{(g)}(s_0) f_2(s_0)-f_1(s_0) f_2^{(g)}(s_0)}{(f_2(s_0))^2}$.

Proof: Note that $f_1(s) \to f_1(s_0)$ and $f_2(s) \to f_2(s_0)$ as $g(s) \to g(s_0)$. Note further that every neighbourhood of $g(s_0)$ contains infinitely many $g(s)$. Also, if $f_2(s_0) \neq 0$, there is a $\delta > 0$ such that $f_2(s) \neq 0$ whenever $|g(s) - g(s_0)| < \delta$ in $g(S)$. With these observations, one can verify the proposition by using the following relations.

$$\frac{(f_1+f_2)(s)-(f_1+f_2)(s_0)}{g(s)-g(s_0)} = \frac{f_1(s)-f_1(s_0)}{g(s)-g(s_0)} + \frac{f_2(s)-f_2(s_0)}{g(s)-g(s_0)}.$$
$$\frac{(cf_1)(s)-(cf_1)(s_0)}{g(s)-g(s_0)} = c\,\frac{f_1(s)-f_1(s_0)}{g(s)-g(s_0)}.$$
$$\frac{(f_1f_2)(s)-(f_1f_2)(s_0)}{g(s)-g(s_0)} = f_1(s)\left(\frac{f_2(s)-f_2(s_0)}{g(s)-g(s_0)}\right) + f_2(s_0)\left(\frac{f_1(s)-f_1(s_0)}{g(s)-g(s_0)}\right).$$
$$\frac{(f_1/f_2)(s)-(f_1/f_2)(s_0)}{g(s)-g(s_0)} = \frac{1}{f_2(s)f_2(s_0)}\left[f_2(s_0)\left(\frac{f_1(s)-f_1(s_0)}{g(s)-g(s_0)}\right) - f_1(s_0)\left(\frac{f_2(s)-f_2(s_0)}{g(s)-g(s_0)}\right)\right].$$

Definition: Let $g: S \to \mathbb{R}$ be any given function. Let $f_n: S \to \mathbb{R}, n = 1, 2, 3, \ldots$ be a sequence of g-defined functions. Let $f: S \to \mathbb{R}$ be another g-defined function. Let $s_0 \in S$ be a fixed point. The sequence of functions f_n is said to converge in quotient to f at s_0 or at $g(s_0)$, relative to g, if there is a neighbourhood U of $g(s_0)$ such that for a given $\varepsilon > 0$ there is an integer n_0 such that $\left|\frac{f_n(s)-f_n(s_0)}{g(s)-g(s_0)} - \frac{f(s)-f(s_0)}{g(s)-g(s_0)}\right| < \varepsilon, \forall n \geq n_0$ whenever $g(s_0) \neq g(s) \in U$.

Proposition: Let $g: S \to \mathbb{R}$ be any given function such that every neighbourhood of every point in $g(S)$ contains infinitely many points of $g(S)$. Let $f_n: S \to \mathbb{R}, n = 1, 2, 3, \ldots$ be a sequence of g-defined functions. Let $f: S \to \mathbb{R}$ be another g-defined function. Let $s_0 \in S$ be a fixed point. Suppose that the sequence of functions f_n converges in quotient to f at s_0 or at $g(s_0)$, relative to g. Suppose that $f_n^{(g)}(s_0)$ exists, for every n. Suppose that the sequence $f_n^{(g)}(s_0), n = 1, 2, 3, \ldots$ converges to a real number A. Then $f^{(g)}(s_0)$ exists and $f_n^{(g)}(s_0) \to f^{(g)}(s_0)$, as $n \to \infty$.

Proof: Let U be a neighbourhood of $g(s_0)$ such that for a given $\varepsilon > 0$ there is an integer n_0 such that $\left|\frac{f_n(s)-f_n(s_0)}{g(s)-g(s_0)} - \frac{f(s)-f(s_0)}{g(s)-g(s_0)}\right| < \varepsilon, \forall n \geq n_0$ whenever $g(s_0) \neq g(s) \in U$. Let us fix $\varepsilon > 0$ and the corresponding n_0 having this property. Suppose further that the integer n_0 has been chosen such that $\left|f_n^{(g)}(s_0) - A\right| < \varepsilon, \forall n \geq n_0$. There is a $\delta > 0$ such that $\left|\frac{f_{n_0}(s)-f_{n_0}(s_0)}{g(s)-g(s_0)} - f_{n_0}^{(g)}(s_0)\right| < \varepsilon$ and such that $g(s) \in U$ whenever $0 < |g(s) - g(s_0)| < \delta$. Now, $\left|\frac{f(s)-f(s_0)}{g(s)-g(s_0)} - A\right| \leq \left|f_{n_0}^{(g)}(s_0) - A\right| + \left|\frac{f_{n_0}(s)-f_{n_0}(s_0)}{g(s)-g(s_0)} - f_{n_0}^{(g)}(s_0)\right| + \left|\frac{f_{n_0}(s)-f_{n_0}(s_0)}{g(s)-g(s_0)} - \frac{f(s)-f(s_0)}{g(s)-g(s_0)}\right| \leq \varepsilon + \varepsilon + \varepsilon$ whenever $g(s) \neq g(s_0)$ and $|g(s) - g(s_0)| < \delta$. This proves the proposition.

The conditions imposed in the statement of the previous proposition are satisfied by the power series functions of the type $f(s) = \sum_{n=0}^{\infty} a_n (g(s))^n, \forall s \in S$, which was discussed in an earlier proposition and therefore the previous proposition can be applied while proving the result corresponding to those power series functions.

Proposition: Let $g: S \to \mathbb{R}$ be any given function. Suppose $f_1: S \to \mathbb{R}$ is continuous on S, relative to g. Suppose $f_1^{(g)}(s_0)$ exists at some point $s_0 \in S$. Suppose that a real valued function f_2 is defined on an interval I which contains $f(S)$, and that f_2 is differentiable at $f_1(s_0)$. If $h(s) = f_2\big(f_1(s)\big), \forall s \in S$, then h is differentiable at the point s_0, relative to g, and $h^{(g)}(s_0) = f_2'\big(f_1(s_0)\big)f_1^{(g)}(s_0)$, where f_2' denotes the usual derivative of f_2.

Proof: Note that h is a g-defined function on S. Let $t_0 = f_1(s_0)$. Then $f_1(s) - f_1(s_0) = \big(g(s) - g(s_0)\big)\left(f_1^{(g)}(s_0) + u(s)\right)$ and $f_2(t) - f_2(t_0) = (t - t_0)(f_2'(t_0) + v(t))$, where $s \in S, t \in I$, $u(s) \to 0$ as $g(s) \to g(s_0)$, and as $v(t) \to 0$ as $t \to t_0$. Let $t = f_1(s)$. Then $t \to t_0$ as $g(s) \to g(s_0)$, and $h(s) - h(s_0) = f_2\big(f_1(s)\big) - f_2\big(f_1(s_0)\big) = \big(f_1(s) - f_1(s_0)\big)(f_2'(t_0) + v(t)) = \big(g(s) - g(s_0)\big)\left(f_1^{(g)}(s_0) + u(s)\right)(f_2'(t_0) + v(t))$. Thus, $\frac{h(s)-h(s_0)}{g(s)-g(s_0)} = \left(f_1^{(g)}(s_0) + u(s)\right)(f_2'(t_0) + v(t))$. Let $g(s) \to g(s_0)$ to conclude that $h^{(g)}(s_0)$ exists and that $h^{(g)}(s_0) = f_1^{(g)}(s_0)f_2'(t_0) = f_1^{(g)}(s_0)f_2'\big(f_1(s_0)\big)$.

Definition: Let $g: S \to \mathbb{R}$ be any given function. Consider a g-defined function $f: S \to \mathbb{R}$. The function f is said to be a non decreasing function relative to g, if $f(s_1) \geq f(s_2)$ whenever $g(s_1) \geq g(s_2)$ with $s_1, s_2 \in S$. Similarly, one can define non increasing functions, strictly increasing functions and strictly decreasing functions relative to g. The function f is said to have a local maximum at s_0 relative to g, if there is an $\varepsilon > 0$ such that if $f(s) \leq f(s_0)$ whenever $g(s_0) - \varepsilon < g(s) < g(s_0) + \varepsilon$ in $g(S)$. Similarly, one can define local minimum (g-local minimum) at some point relative to g.
The next result is about derivative values for local maximum points and local minimum points relative to a function.

Proposition: Let $g: S \to \mathbb{R}$ be a given function on a set S. Suppose $f: S \to \mathbb{R}$ be a g-defined function such that $f^{(g)}(s_0)$ exists and a g-local maximum exists for f at s_0, for some point $s_0 \in S$. Suppose further that for every $\varepsilon > 0$, the intervals $\big(g(s_0) - \varepsilon, g(s_0)\big)$ and $(g(s_0), g(s_0) + \varepsilon)$ contain infinitely many points of $g(S)$. Then $f^{(g)}(s_0) = 0$.

Proof: Fix $\varepsilon > 0$ such that $f(s) \leq f(s_0)$ whenever $g(s_0) - \varepsilon < g(s) < g(s_0)$ and whenever $g(s_0) < g(s) < g(s_0) + \varepsilon$. Then $\frac{f(s)-f(s_0)}{g(s)-g(s_0)} \leq 0$ whenever $g(s_0) < g(s) < g(s_0) + \varepsilon$, and $\frac{f(s)-f(s_0)}{g(s)-g(s_0)} \geq 0$ whenever $g(s_0) - \varepsilon < g(s) < g(s_0)$. If $g(s)$ approaches $g(s_0)$ from right side of $g(s_0)$, then it is concluded that $f^{(g)}(s_0) \leq 0$, and if $g(s)$ approaches $g(s_0)$ from left side of $g(s_0)$, then it is concluded that $f^{(g)}(s_0) \geq 0$. So, $f^{(g)}(s_0) = 0$.
To avoid the assumptions of the type "for every $\varepsilon > 0$, the intervals $\big(g(s_0) - \varepsilon, g(s_0)\big)$ and $(g(s_0), g(s_0) + \varepsilon)$ contain infinitely many points of $g(S)$", it is to be assumed in the remaining part that $g(S)$ contains intervals.

Convention: Let f be a g-defined function on a set S. Then $f\big(g^{-1}(y)\big)$ or $f\big(g^{-1}(\{y\})\big)$ is a singleton subset of $f(S)$, for any $s \in S$. These singleton subsets will be considered as single elements of $f(S)$.

Rolle's mean value theorem: Let $g: S \to \mathbb{R}$ be a given function on a set S such that $[a,b] \subseteq g(S)$ for some real numbers a and b satisfying $a < b$. Suppose $f: S \to \mathbb{R}$ is continuous on $[a,b]$ relative to g, and f is differentiable on (a,b) relative to g. If $f\left(g^{-1}(a)\right) = f\left(g^{-1}(b)\right)$, then there is a $s_0 \in S$ such that $f^{(g)}(s_0) = 0$ and such that $a < g(s_0) < b$.

Proof: If $f\left(g^{-1}(a)\right) = f\left(g^{-1}(b)\right) = f(s)$ whenever $a < g(s) < b$, then $f^{(g)}(s) = 0$, whenever $a < g(s) < b$. If $f(s) > f\left(g^{-1}(a)\right) = f\left(g^{-1}(b)\right)$, for some $s \in S$ satisfying $a < g(s) < b$, then there is a $s_0 \in S$ such that $a < g(s_0) < b$ and such that $f(s) \leq f(s_0)$ whenever $g(s) \in [a,b]$. In this case $f^{(g)}(s_0) = 0$, by the previous proposition. If $f(s) < f\left(g^{-1}(a)\right) = f\left(g^{-1}(b)\right)$, for some $s \in S$ satisfying $a < g(s) < b$, then there is a $s_0 \in S$ such that $a < g(s_0) < b$ and such that $f(s) \geq f(s_0)$ whenever $g(s) \in [a,b]$. Again, $f^{(g)}(s_0) = 0$, by the previous proposition. In all cases, there is a $s_0 \in S$ such that $f^{(g)}(s_0) = 0$ and such that $a < g(s_0) < b$.

Cauchy's mean value theorem: Let $g: S \to \mathbb{R}$ be a given function on a set S such that $[a,b] \subseteq g(S)$ for some real numbers a and b satisfying $a < b$. Suppose $f_1: S \to \mathbb{R}$ and $f_2: S \to \mathbb{R}$ are continuous on $[a,b]$ relative to g, and f_1 and f_2 are differentiable on (a,b) relative to g. Suppose $f_2\left(g^{-1}(a)\right) \neq f_2\left(g^{-1}(b)\right)$ and $f_2^{(g)}(s) \neq 0$, whenever $a < g(s) < b$. Then there is a $s_0 \in S$ such that $a < g(s_0) < b$ and such that $\frac{f_1\left(g^{-1}(b)\right) - f_1\left(g^{-1}(a)\right)}{f_2\left(g^{-1}(b)\right) - f_2\left(g^{-1}(a)\right)} = \frac{f_1^{(g)}(s_0)}{f_2^{(g)}(s_0)}$.

Proof: Let $h(s) = \left[f_1\left(g^{-1}(b)\right) - f_1\left(g^{-1}(a)\right)\right]f_2(s) - \left[f_2\left(g^{-1}(b)\right) - f_2\left(g^{-1}(a)\right)\right]f_1(s)$. Then $h\left(g^{-1}(b)\right) = f_1\left(g^{-1}(b)\right)f_2\left(g^{-1}(a)\right) - f_1\left(g^{-1}(a)\right)f_2\left(g^{-1}(b)\right) = h\left(g^{-1}(a)\right)$. By the Rolle's mean value theorem, there is a $s_0 \in S$ such that $a < g(s_0) < b$ and such that $h^{(g)}(s_0) = 0$. That is, $\frac{f_1\left(g^{-1}(b)\right) - f_1\left(g^{-1}(a)\right)}{f_2\left(g^{-1}(b)\right) - f_2\left(g^{-1}(a)\right)} = \frac{f_1^{(g)}(s_0)}{f_2^{(g)}(s_0)}$.

Lagrange's mean value theorem: Let $g: S \to \mathbb{R}$ be a given function on a set S such that $[a,b] \subseteq g(S)$ for some real numbers a and b satisfying $a < b$. Suppose $f: S \to \mathbb{R}$ is continuous on $[a,b]$ relative to g, and f is differentiable on (a,b) relative to g. Then there is a $s_0 \in S$ such that $a < g(s_0) < b$ and such that $\frac{f\left(g^{-1}(b)\right) - f\left(g^{-1}(a)\right)}{b-a} = f^{(g)}(s_0)$.

Proof: Take $f_1 = f$ and $f_2 = g$ in the previous theorem or take $h(s) = \left[f\left(g^{-1}(b)\right) - f\left(g^{-1}(a)\right)\right]g(s) - [b-a]f(s)$ in the proof of the previous theorem.

L'Hospital's rule: Let $g: S \to \mathbb{R}$ be a given function on a set S such that $(a, b) \subseteq g(S)$ for some a and b satisfying $-\infty \leq a < b \leq \infty$. Suppose $f_1: S \to \mathbb{R}$ and $f_2: S \to \mathbb{R}$ are functions differentiable on (a, b) relative to g. Suppose $f_2^{(g)}(s) \neq 0 \neq f_2(s)$, whenever $a < g(s) < b$. Suppose $f_2(s_1) \neq f_2(s_2)$ whenever $a < g(s_1) < g(s_2) < b$. Suppose further that $\frac{f_1^{(g)}(s)}{f_2^{(g)}(s)} \to A$ as $g(s) \to a$. Assume further that

(I) $f_1(s) \to 0$ and $f_2(s) \to 0$ as $g(s) \to a$

or

(II) $f_2(s) \to +\infty$ as $g(s) \to a$.

Then $\frac{f_1(s)}{f_2(s)} \to A$ as $g(s) \to a$.

Proof: Let us first assume that $-\infty \leq A < \infty$. Choose a real number q such that $A < q$. Choose another real number r such that $A < r < q$. Then there is a point $c \in (a, b)$ such that $\frac{f_1^{(g)}(s)}{f_2^{(g)}(s)} < r$, whenever $a < g(s) < c$. If $a < g(s_1) < g(s_2) < c$, then there is a point $g(t) \in \left(g(s_1), g(s_2)\right)$ such that

(III) $\frac{f_1(s_1)-f_1(s_2)}{f_2(s_1)-f_2(s_2)} = \frac{f_1^{(g)}(t)}{f_2^{(g)}(t)} < r.$

Suppose now (I) holds. Letting $g(s_1) \to a$, (III) implies that $\frac{f_1(s_2)}{f_2(s_2)} \leq r < q$ when $a < g(s_2) < c$. Suppose now (II) holds. Keeping s_2 fixed, let us choose $c_1 \in \left(a, g(s_2)\right)$ such that $f_2(s_1) > f_2(s_2)$ and $f_2(s_1) > 0$ when $a < g(s_1) < c_1$. Multiply (III) by $\frac{f_2(s_1)-f_2(s_2)}{f_2(s_1)}$ and get $\frac{f_1(s_1)}{f_2(s_1)} < r - r\frac{f_2(s_2)}{f_2(s_1)} + \frac{f_1(s_2)}{f_2(s_1)}$ whenever $a < g(s_1) < c_1$. Then there is a point $c_2 \in (a, c_1)$ such that $\frac{f_1(s)}{f_2(s)} < q$, whenever $0 < g(s) < c_2$ and whenever $A < q$.

Let us now assume that $-\infty < A \leq +\infty$. Choose a point p such that $p < A$ and find c_3 such that $p < \frac{f_1(s)}{f_2(s)}$ for $a < g(s) < c_3$. The result follows on combining all of them.

A generalized version of the Lagrange's mean value theorem is the following Taylor's theorem. Let us use the notation $f^{(g)^2}(s_0)$ for the second order derivative of f at s_0 relative to g, that is, the derivative of $f^{(g)}$ at s_0 relative to g, when it exists in the usual meaningful way. In general, let us use the notation $f^{(g)^n}(s_0)$ for the n-th order derivative of f at s_0 relative to g, for any natural number n, with a convention that it is $f(s_0)$ for the value $n = 0$.

Taylor's theorem: Let $g: S \to \mathbb{R}$ be a given function on a set S such that $[a, b] \subseteq g(S)$ for some real numbers a and b satisfying $a < b$. Let n be any natural number. Suppose $f: S \to \mathbb{R}$ is a g-defined function such that $f^{(g)}, f^{(g)^2}, \dots, f^{(g)^{n-1}}$ exist and they are g-continuous on $[a, b]$, and such that $f^{(g)^n}$ exist and g-continuous on (a, b). Let α, β be two distinct point of $[a, b]$. Let us use the notations α_s, β_s for points in $[a, b]$ satisfying $g(\alpha_s) = \alpha, g(\beta_s) = \beta$. Define $P(t) = \sum_{k=0}^{n-1} \frac{f^{(g)^k}(\alpha_s)}{k!}(t - \alpha)^k$. Then there is a point x between α and β such that $f(\beta_s) = P(\beta) + \frac{f^{(g)^n}(x_s)}{n!}(\beta - \alpha)^n$ when $g(x_s) = x$.

Proof:

(I) Let M be the number defined by $f(\beta) = P(\beta) + M(\beta - \alpha)^n$.

(II) Write $f_1(t) = f(t) - P(t) - M(t - \alpha)^n$, when $a \leq t \leq b$.

By (I) and (II), it is true that

(III) $f_1^{(g)^n}(t) = f^{(g)^n}(t) - n!\,M$, when $a < t < b$.

Since $P^{(g)^k}(\alpha) = f^{(g)^k}(\alpha)$ for $k = 0, 1, 2, \dots, n-1$, then $f_1(\alpha) = f_1^{(g)}(\alpha) = f_1^{(g)^2}(\alpha) = \cdots = f_1^{(g)^{n-1}}(\alpha) = 0$. By (I) and (II), $f_1(\beta) = 0$. Therefore $f_1^{(g)}(x_1) = 0$ for some x_1 between α and β satisfying $\alpha \neq x_1 \neq \beta$. Since $f_1^{(g)}(\alpha) = 0$, there is a x_2 between α and x_1 ($\alpha \neq x_2 \neq x_1$) such that $f_1^{(g)^2}(x_2) = 0$. After n steps, it is concluded that $f_1^{(g)^n}(x_n) = 0$ for some x_n between α and x_{n-1}, so that x_n is between α and β satisfying $\alpha \neq x_n \neq \beta$. In this case, by (I), $f^{(g)^n}(x_n) = n!\,M = n!\left[\frac{f(\beta)-P(\beta)}{(\beta-\alpha)^n}\right]$, or, $f(\beta_s) = P(\beta) + \frac{f^{(g)^n}(x_n)}{n!}(\beta - \alpha)^n$. This proves the theorem.

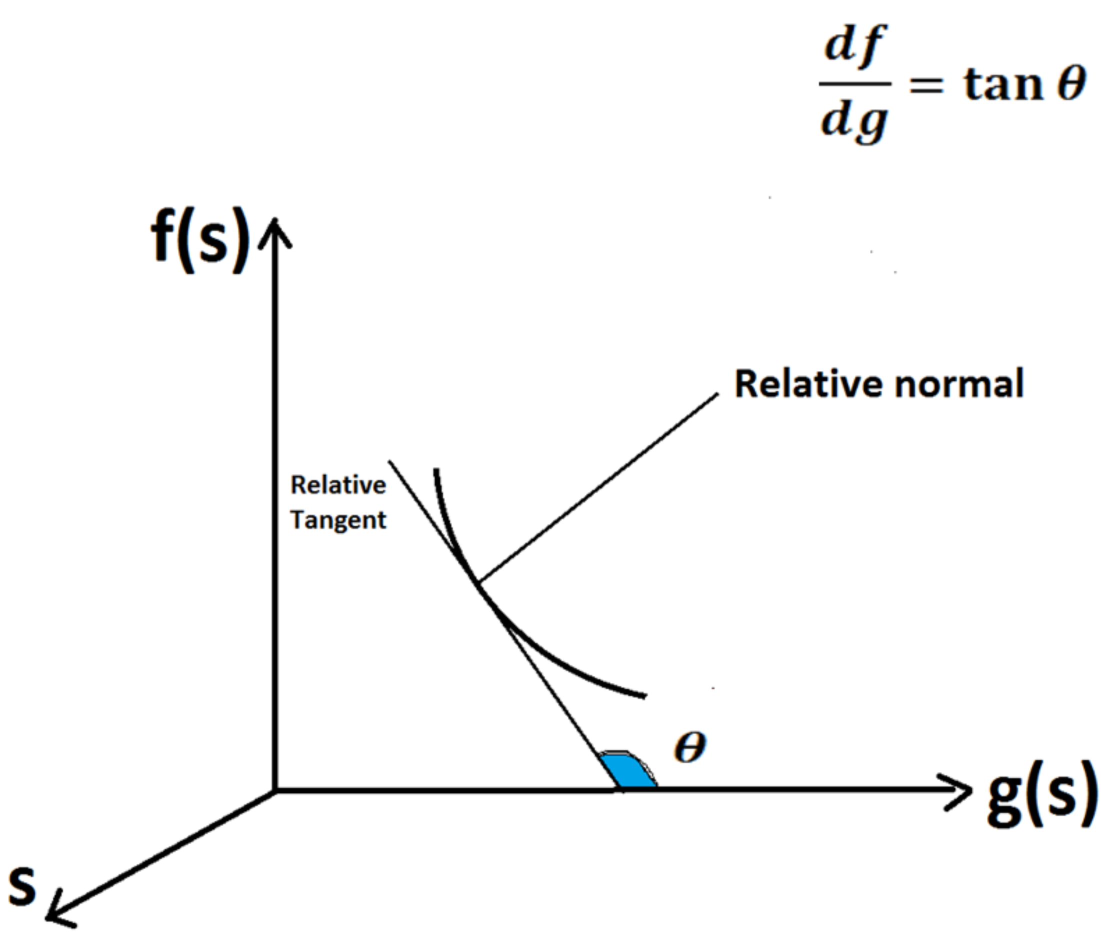

Figure 1. Relative differentiation.

Let us observe Figure 1 for a geometric interpretation of differentiation relative to a function as tan of the angle made by the curve for f relative to g. Let us always observe that arguments for proofs can never be based directly on such figures.

5. INTEGRATION WITH RESPECT TO A FUNCTION

Let $g: S \to \mathbb{R}$ be a given function on a set S such that $[a,b] \subseteq g(S)$ for some $a < b$. Consider a g-defined function $f: S \to \mathbb{R}$ such that $f\left(g^{-1}([a,b])\right)$ is a bounded set. Let us say in this case that f is g-bounded on $[a,b]$. Let $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a,b]$ in the sense that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. For each $i = 1,2,\dots,n$, let $M_i = Sup_{x_{i-1} \le g(s) \le x_i} f(s)$ and $m_i = Inf_{x_{i-1} \le g(s) \le x_i} f(s)$. Define $L_g(\mathcal{P}, f) = \sum_{1=1}^{n} m_i(x_i - x_{i-1})$ and $U_g(\mathcal{P}, f) = \sum_{1=1}^{n} M_i(x_i - x_{i-1})$. Let $\int_{a\_}^{b} f\,dg = Sup_{\mathcal{P}} L_g(\mathcal{P}, f)$ and $\int_{a}^{b-} f\,dg = Inf_{\mathcal{P}} U_g(\mathcal{P}, f)$. A partition $\mathcal{P}^*$ of $[a,b]$ is called a refinement of another partition $\mathcal{P}$ of $[a,b]$, if $\mathcal{P} \subseteq \mathcal{P}^*$. Let us follow these notations and definition in the sequel. A visualization may helpful with the help of Figure 2.

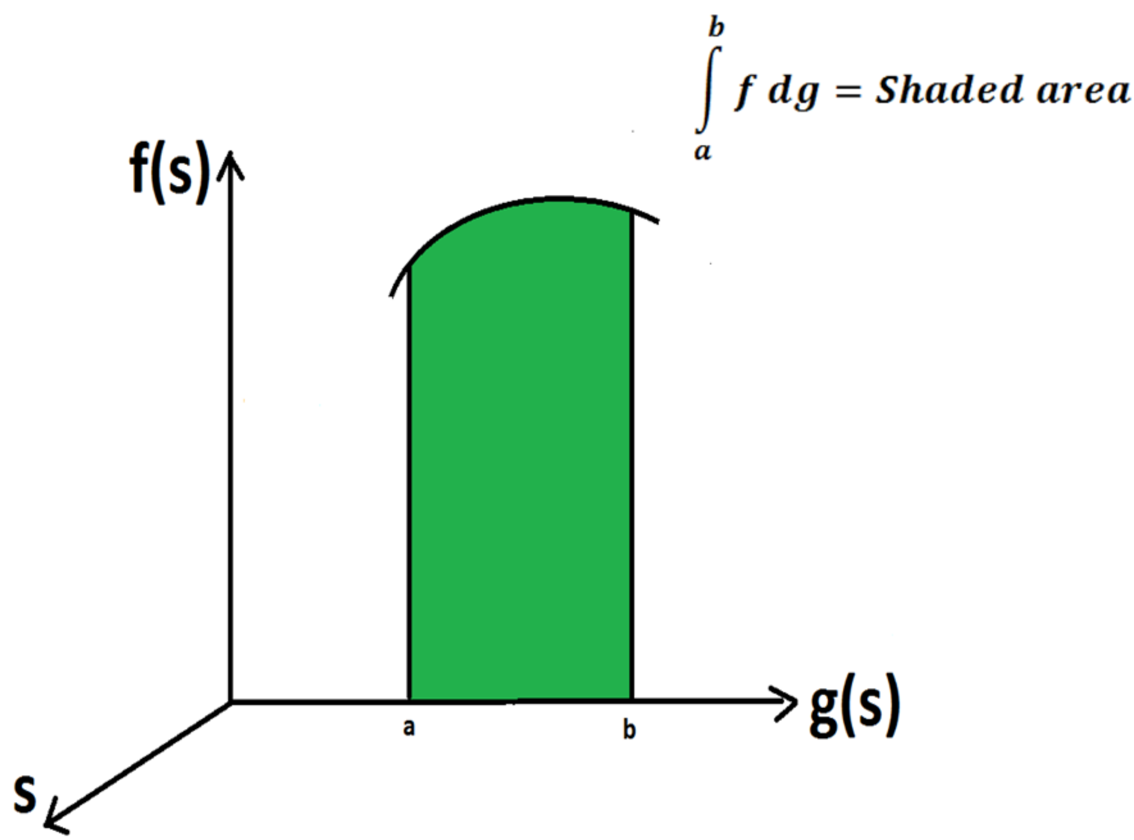

Figure 2. Relative integration.

Proposition: If $\mathcal{P}^*$ is a refinement of $\mathcal{P}$ of $[a,b]$, then (i) $L_g(\mathcal{P},f) \leq L_g(\mathcal{P}^*,f)$, and (ii) $U_g(\mathcal{P}^*,f) \leq U_g(\mathcal{P},f)$.

Proof: It is enough to prove the result for the case that $\mathcal{P}^*$ contains just one more point than the points in $\mathcal{P}$. Suppose that x^* is the extra point and that $x_{i-1} < x^* < x_i$. Write $w_1 = Inf_{x_{i-1} \leq g(s) \leq x^*} f(s)$, $w_2 = Inf_{x^* \leq g(s) \leq x_i} f(s)$ and $m_i = Inf_{x_{i-1} \leq g(s) \leq x_i} f(s)$. Then $L_g(\mathcal{P}^*,f) - L_g(\mathcal{P},f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1}) = (w_1 - m_i)(x^* - x_{i-1}) + (w_2 - m_i)(x_i - x^*) \geq 0$. This proves the part (i). The second part (ii) can be proved in a similar way.

Corollary: $\int_{a\_}^{b} f\,dg \leq \int_{a}^{b-} f\,dg$.

Definition: Let us write $f \in \mathcal{R}_g$ over $[a,b]$, when $\int_{a\_}^{b} f\,dg = \int_{a}^{b-} f\,dg$, and let us write $\int_a^b f\,dg$ for this common value. Let us say in this case that f is Riemann integrable over $[a,b]$, with respect to g, and let us call $\int_a^b f\,dg$ as the Riemann integral of f over $[a,b]$ with respect to g.

Proposition: $f \in \mathcal{R}_g$ over $[a,b]$ if and only if for every $\varepsilon > 0$, there is a partition $\mathcal{P}$ of $[a,b]$ such that: (I) $U_g(\mathcal{P},f) - L_g(\mathcal{P},f) < \varepsilon$.

Proof: Suppose (I) holds. Then $L_g(\mathcal{P},f) \leq \int_{a\_}^{b} f\,dg \leq \int_{a}^{b-} f\,dg \leq U_g(\mathcal{P},f)$ implies that $0 \leq \int_{a}^{b-} f\,dg - \int_{a\_}^{b} f\,dg < \varepsilon$. Since this is true for every $\varepsilon > 0$, it is concluded that $\int_{a\_}^{b} f\,dg = \int_{a}^{b-} f\,dg$ so that $f \in \mathcal{R}_g$ over $[a,b]$.

Conversely assume that $f \in \mathcal{R}_g$ over $[a,b]$. Let $\varepsilon > 0$ be given. Then there is a common partition $\mathcal{P}$ for $[a,b]$ such that $U_g(\mathcal{P},f) - \int_{a}^{b-} f\,dg < \frac{\varepsilon}{2}$ and $\int_{a\_}^{b} f\,dg - L_g(\mathcal{P},f) < \frac{\varepsilon}{2}$. In this case $U_g(\mathcal{P},f) - L_g(\mathcal{P},f) < \varepsilon$.

Remark: If $U_g(\mathcal{P},f) - L_g(\mathcal{P},f) < \varepsilon$, then for any partition $\mathcal{P}^*$, which is a refinement of $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n\}$, then $U_g(\mathcal{P}^*,f) - L_g(\mathcal{P}^*,f) \leq U_g(\mathcal{P},f) - L_g(\mathcal{P},f) < \varepsilon$, $\left|\sum_{i=1}^{n} f(s_i)[x_i - x_{i-1}] - \int_{a\_}^{b} f\,dg\right| < \varepsilon$ and $\left|\sum_{i=1}^{n} f(s_i)[x_i - x_{i-1}] - \int_{a}^{b-} f\,dg\right| < \varepsilon$, for any s_i satisfying $x_{i-1} \leq g(s_i) \leq x_i$, when $m_i \leq f(s_i) \leq M_i$. So, the following proposition is true.

Proposition: $f \in \mathcal{R}_g$ over $[a,b]$ if and only if there is a common real number α such that for any given $\varepsilon > 0$, there is a partition $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ such that $\left|\sum_{i=1}^{n} f(s_i)[x_i - x_{i-1}] - \alpha\right| < \varepsilon$, for any s_i satisfying $x_{i-1} \leq g(s_i) \leq x_i$. In this case, the common number $\alpha = \int_a^b f\,dg$.

Proposition: If f is (uniformly) continuous on $[a,b]$ relative to g, then $f \in \mathcal{R}_g$ over $[a,b]$.

Proof: Let $\varepsilon > 0$ be given. Let us choose $\eta > 0$ such that $(b-a)\eta < \varepsilon$. Since f is uniformly continuous, there is a $\delta > 0$ such that $|f(s_1) - f(s_2)| < \eta$, whenever $|g(s_1) - g(s_2)| < \delta$ and $g(s_1), g(s_2) \in [a,b]$. If $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ is a partition of $[a,b]$ such that $x_i - x_{i-1} < \delta, \forall i$, then $M_i - m_i \leq \eta, \forall i$ so that $U_g(\mathcal{P}, f) - L_g(\mathcal{P}, f) \leq \sum_{i=1}^{n} \eta[x_i - x_{i-1}] = \eta(b-a) < \varepsilon$. This proves the result.

Proposition: If f is continuous on $[a,b]\backslash A$, relative to g, when A is a finite set, then $f \in \mathcal{R}_g$.

Proof: Cover the set A by means of finitely many open intervals whose total length is less than any desired positive number. Include the end points of these intervals as two successive points in a partition $\mathcal{P}$ considered for $U_g(\mathcal{P}, f) - L_g(\mathcal{P}, f)$ in the previous proof. A modified argument of the previous proof may be considered for the complement of the union of these open intervals along with g-boundedness of f on the open intervals. Such a modification leads to a proof.

Proposition: Suppose $f \in \mathcal{R}_g$ over $[a,b]$ and $m \leq f(s) \leq M$ whenever $a \leq g(s) \leq b$. Suppose φ is a continuous real valued function on $[m,M]$. Let $h(s) = \varphi(f(s))$ whenever $a \leq g(s) \leq b$. Then $h \in \mathcal{R}_g$ on $[a,b]$.

Proof: Let $\varepsilon > 0$ be given. Since φ is uniformly continuous on $[m,M]$, there exists a $\delta > 0$ such that $\delta < \varepsilon$ and such that $|\varphi(s) - \varphi(t)| < \varepsilon$ whenever $|s-t| \leq \delta$ and $s,t \in [m,M]$. Since $f \in \mathcal{R}_g$, there is a partition $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a,b]$ such that $U_g(\mathcal{P}, f) - L_g(\mathcal{P}, f) < \delta^2$. Let $m_i = Inf_{x_{i-1} \leq g(s) \leq x_i} f(s), m_i^* = Inf_{x_{i-1} \leq g(s) \leq x_i} h(s)$, and $M_i = Sup_{x_{i-1} \leq g(s) \leq x_i} f(s), M_i^* = Sup_{x_{i-1} \leq g(s) \leq x_i} h(s)$, for every i. Divide the numbers $i = 1, 2, \dots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$; $i \in B$ if $M_i - m_i \geq \delta$. For $i \in A$, $M_i^* - m_i^* \leq \varepsilon$. For $i \in B$, $M_i^* - m_i^* \leq 2K$, where $K = Sup\,\{|\varphi(t)|: m \leq t \leq M\}$. Then $\delta \sum_{i \in B}(x_i - x_{i-1}) \leq \sum_{i \in B}(M_i - m_{i-1})\,(x_i - x_{i-1}) < \delta^2$ so that $\sum_{i \in B}(x_i - x_{i-1}) < \delta$. It follows now that $U_g(\mathcal{P}, h) - L_g(\mathcal{P}, h) = \sum_{i \in A}(M_i^* - m_i^*)(x_i - x_{i-1}) + \sum_{i \in B}(M_i^* - m_i^*)(x_i - x_{i-1}) \leq \varepsilon(b-a) + 2K\delta < \varepsilon(b-a+2K)$.

Corollary: If $f \in \mathcal{R}_g$, then $f^2 \in \mathcal{R}_g$ and $|f| \in \mathcal{R}_g$ on $[a,b]$.

Remark: The followings are true by considering a common interval $[a,b]$ of $g(S)$.

(a) If $f_1, f_2 \in \mathcal{R}_g$, then $f_1 + f_2 \in \mathcal{R}_g$ and $\lambda f \in \mathcal{R}_g$ for any real number λ. Also $\int_a^b (f_1 + f_2)\, dg = \int_a^b f_1\, dg + \int_a^b f_2\, dg$ and $\int_a^b (\lambda f_1)\, dg = \lambda \int_a^b f_1\, dg$.

(b) If $f_1, f_2 \in \mathcal{R}_g$ and if $f_1 \leq f_2$ on $[a,b]$, then $\int_a^b f_1\, dg \leq \int_a^b f_2\, dg$.

(c) If $f \in \mathcal{R}_g$ on $[a,b]$, then $f \in \mathcal{R}_g$ on $[a,c]$ and $f \in \mathcal{R}_g$ on $[c,b]$, for any $c \in (a,b)$.

(d) If $f \in \mathcal{R}_g$ and $|f| \leq M$ on $[a,b]$, then $\left|\int_a^b f\, dg\right| \leq M(b-a)$.

(e) If $f_1, f_2 \in \mathcal{R}_g$, then $f_1 f_2 \in \mathcal{R}_g$, because $4f_1 f_2 = (f_1 + f_2)^2 - (f_1 - f_2)^2$.

(f) If $f \in \mathcal{R}_g$, then $\left|\int_a^b f\, dg\right| \leq \int_a^b |f|\, dg$. For, let $c = \pm 1$ be such that $c \int_a^b f\, dg \geq 0$. Then $\left|\int_a^b f\, dg\right| = c \int_a^b f\, dg = \int_a^b cf\, dg \leq \int_a^b |f|\, dg$.

Proposition: Let $f \in \mathcal{R}_g$ on $[a, b]$. For $a \le x \le b$, and for $g(t) = x$, let $F(t) = \int_a^x f(s)dg(s)$. Then F is continuous relative to g on $[a, b]$. If f is continuous at a point s_0 with $a \le g(s_0) \le b$, relative to g, then F is differentiable at s_0, relative to g, and $F^{(g)}(s_0) = f(s_0)$.

Proof: If $a \le x_1 < x_2 \le b$, $g(t_1) = x_1$, and $g(t_2) = x_2$, then $|F(t_2) - F(t_1)| = \left|\int_{x_1}^{x_2} f(s)dg(s)\right| \le M(x_2 - x_1)$, when $M = Sup_{a \le g(s) \le b}|f(s)|$. So, F is uniformly continuous on $[a, b]$. Now, suppose f is continuous at $x_0 \in [a, b]$ relative to g, and let $g(s_0) = x_0$. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|f(s) - f(s_0)| < \varepsilon$ whenever $|g(s) - g(s_0)| < \delta$ with $g(s), g(s_0) \in [a, b]$. Hence, if $x_0 - \delta < g(s) \le x_0 \le g(t) < x_0 + \delta$, $g(s) \ne g(t)$, $g(s) \in [a, b]$, $g(t) \in [a, b]$, and if $x = g(s)$, $y = g(t)$, then $\left|\frac{F(t)-F(s)}{g(t)-g(s)} - f(x_0)\right| = \left|\frac{\int_x^y (f(u)-f(x_0))\, du}{g(t)-g(s)}\right| \le \varepsilon$. It follows that $F^{(g)}(x_0) = f(x_0)$.

Theorem: If $f \in \mathcal{R}_g$ over $[a, b] \subseteq g(S)$, and if there is a differentiable function F on $[a, b]$ such that $F^{(g)} = f$, then $\int_a^b f(s)dg(s) = F(b) - F(a)$.

Proof: Let $\varepsilon > 0$ be given. Choose a partition $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that $U_g(\mathcal{P}, f) - L_g(\mathcal{P}, f) < \varepsilon$. The Lagrange mean value theorem furnishes points $y_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = f(t_i)[x_i - x_{i-1}]$. Thus, $\sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}] = F(b) - F(a)$. Therefore, $\left|F(b) - F(a) - \int_a^b f(s)dg(s)\right| \le \varepsilon$, for every $\varepsilon > 0$.

6. OTHER DIRECTIONS

An initial value problem takes the form: Find a g-function $f: S \to \mathbb{R}$ satisfying $\frac{df(s)}{dg(s)} = F(g(s), f(s))$ and satisfying $f(s_0) = c$ (or, $f\big(g(s_0)\big) = c$). Here $g: S \to \mathbb{R}$ is a given function, $f: S \to \mathbb{R}$ is an unknown g-function that needs to be found, $F: (g(S) \times \mathbb{R}) \to \mathbb{R}$ is a given function, $s_0 \in S$ is a given point, and c is a given constant. The following is the equivalent integral equation problem. Find a g-function $f: S \to \mathbb{R}$ satisfying $f(s) = c + \int_{g(s_0)}^{g(s)} F(g(t), f(t))\, dg(t)$. The following relation provides Picard's iteration formula. $f_{n+1}(s) = c + \int_{g(s_0)}^{g(s)} F(g(t), f_n(t))\, dg(t)$, for $n = 0, 1, 2, \dots$, with $f_0(s) = c, \forall s$. The Euler's method is given by $f(s_{i+1}) = f(s_i) + hF\big(g(s_i), f(s_i)\big)$ when $g(s_{i+1}) = g(s_i) + h$, $i = 0, 1, 2, \dots$, is satisfied with $h > 0$. One may try to derive an existence of solution to the initial value problem under suitable conditions, by using Euler's method and by using Picard's iteration method.

The following relative concepts can be considered in connection with partial differentiation, partial integration and joint integration. Let $S = S_1 \times S_2 \times \ldots \times S_m$, $X = X_1 \times X_2 \times \ldots \times X_n$ and $Y = Y_1 \times Y_2 \times \ldots \times Y_m$. Consider $g: S \to Y$ in the form $g(s_1, s_2, \ldots, s_m) = \big(g_1(s_1), g_2(s_2), \ldots, g_m(s_m)\big)$ with $g_i: S \to Y_i$, for $i = 1, 2, \ldots, m$. Consider $f: S \to X$ in the form $f(s_1, s_2, \ldots, s_m) = \big(f_1(s), f_2(s), \ldots, f_n(s)\big)$ with $f_j: S \to X_j$ for $j = 1, 2, \ldots, n$ and $s = (s_1, s_2, \ldots, s_m)$. Suppose f is a g-defined function. That is, if $s_1 = (s_{11}, s_{12}, \ldots, s_{1m})$, $s_2 = (s_{21}, s_{22}, \ldots, s_{2m})$ in S, and if $g_i(s_{1i}) = g(s_{2i})$, for all $i = 1, 2, \ldots, m$, then $f_j(s_1) = f_j(s_2)$, for all $j = 1, 2, \ldots, n$. If $s_0 = (s_{01}, s_{02}, \ldots, s_{0m})$ and if $g_i(s) = g_i(t)$ for some i, then $g\big(s_{0,1}, s_{0,2}, \ldots s_{0,i-1}, s, s_{0,i+1}, \ldots, s_{0,m}\big) = g\big(s_{0,1}, s_{0,2}, \ldots, s_{0,i-1}, t, s_{0,i+1}, \ldots, s_{0,m}\big)$ so that $f\big(s_{0,1}, s_{0,2}, \ldots s_{0,i-1}, s, s_{0,i+1}, \ldots, s_{0,m}\big) = f\big(s_{0,1}, s_{0,2}, \ldots, s_{0,i-1}, t, s_{0,i+1}, \ldots, s_{0,m}\big)$ and hence $f_j\big(s_{0,1}, s_{0,2}, \ldots s_{0,i-1}, s, s_{0,i+1}, \ldots, s_{0,m}\big) = f_j\big(s_{0,1}, s_{0,2}, \ldots, s_{0,i-1}, t, s_{0,i+1}, \ldots, s_{0,m}\big)$, for every j. Thus, if $X_1 = \cdots = X_n = Y_1 = \cdots = Y_m = \mathbb{R}$, one can define $\frac{\partial f_j}{\partial g_i}\big(s_{0,1}, s_{0,2}, \ldots, s_{0,i-1}, t, s_{0,i+1}, \ldots, s_{0,m}\big)$ as the limit of the ratio $\frac{f(s_{0,1}, s_{0,2}, \ldots s_{0,i-1}, s, s_{0,i+1}, \ldots, s_{0,m}) - f(s_{0,1}, s_{0,2}, \ldots, s_{0,i-1}, t, s_{0,i+1}, \ldots, s_{0,m})}{g_i(s) - g_i(t)}$ as $g_i(s) \to g_i(t)$ with $g_i(s) \neq g_i(t)$.

Suppose $X_1 = \cdots = X_n = Y_1 = \cdots = Y_m = \mathbb{R}$ and $I_i = [a_i, b_i] \subseteq g_i(S_i), \forall i = 1, 2, \ldots, m$, when $a_i < b_i, \forall i$. Suppose further that $S_1 = S_2 = \cdots = S_m$ and $g_1 = g_2 = \cdots = g_m$ so that f_j is a g_i-function $\forall i, j$. Let $I = \prod_{i=1}^{n} I_i$. Partial integration $\int_{a_i}^{b_i} f_j(s_1, \ldots, s_m)\, dg_i$ is obtained as an ordinary g_i-integration by keeping $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m$ as fixed and by varying $g_i(s_i)$ over $[a_i, b_i]$. This partial integration becomes a function of $g_1(s_1), \ldots, g_{i-1}(s_{i-1}), g_{i+1}(s_{i+1}), \ldots, g_m(s_m)$. Now, one can define $\int_I f\, dg$ as an element of $\mathbb{R}^n$ with j-th component as $\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} \ldots \left(\int_{a_{n-1}}^{b_{n-1}}\left(\int_{a_n}^{b_n} f_j\, dg_n\right) dg_{n-1}\right) \ldots dg_2\right) dg_1$ in terms of partial integrations, without referring to measure theory.

References

[1] M.W. Botsko, An elementary proof of Lebesgue's differentiation theorem. *The American Mathematical Monthly 110*(9) (2003) 834-838.

[2] J. Browne, and J.D. Cook, An Intuitive Approach to L'Hospital's Rule. *The Mathematics Teacher 75*(9) (1982) 757-758.

[3] A.M. Bruckner, creating differentiability and destroying derivatives. *The American Mathematical Monthly 85*(7) (1978) 554-562.

[4] D.S. Carter, L'hospital's rule for complex-valued functions. *The American Mathematical Monthly 65*(4) (1958) 264-266.

[5] P.J. Daniell, Differentiation with respect to a function of limited variation. *Transactions of the American Mathematical Society 19*(4) (1918) 353-362.

[6] P.J. Daniell, A general form of integral. *Annals of mathematics 19*(4) (1918) 279-294.

[7] M. De Guzman, A change-of-variables formula without continuity. *The American Mathematical Monthly 87*(9) (1980) 736-739.

[8] P. Dierolf, and V. Schmidt, A proof of the change of variable formula for d-dimensional integrals. *The American mathematical monthly 105*(7) (1998) 654-656.

[9] B. Eisenberg, and R. Sullivan, The fundamental theorem of calculus in two dimensions. *The American mathematical monthly 109*(9) (2002) 806-817.

[10] H. Flanders, Differentiation under the integral sign. *The American Mathematical Monthly 80*(6) (1973) 615-627.

[11] T.M. Flett, A mean value theorem. *The Mathematical Gazette 42*(339) (1958) 38-39.

[12] M. Furi, and M. Martelli, On the mean value theorem, inequality, and inclusion. *The American Mathematical Monthly 98*(9) (1991) 840-846.

[13] D.B. Goodner, Mean value theorems for functions with finite derivates. *The American Mathematical Monthly 67*(9) (1960) 852-855.

[14] E.C. Grigoletto, and E.C. De Oliveira, Fractional versions of the fundamental theorem of calculus. *Applied Mathematics 4*(7) (2013) 23-33.

[15] H.L. Johnson, Investigating the fundamental theorem of calculus. *MatheMatics teacher 103*(6) (2010) 430-435.

[16] J.J. Koliha, A fundamental theorem of calculus for Lebesgue integration. *The American Mathematical Monthly 113*(6) (2006) 551-555.

[17] P.D. Lax, Change of variables in multiple integrals. *The American mathematical monthly 106*(6) (1999) 497-501.

[18] P.D. Lax, Change of variables in multiple integrals II. *The American Mathematical Monthly 108*(2) (2001) 115-119.

[19] W.F. Pfeffer, The multidimensional fundamental theorem of calculus. *Journal of the Australian Mathematical Society 43*(2) (1987) 143-170.

[20] E.L. Post, Generalized differentiation. *Transactions of the American Mathematical Society 32*(4) (1930) 723-781.

[21] S. Reich, On mean value theorems. *The American Mathematical Monthly 76*(1) (1969) 70-73.

[22] J. Schwartz, The formula for change in variables in a multiple integral. *The American Mathematical Monthly 61*(2) (1954) 81-85.

[23] J. Serrin, and D.E. Varberg, A general chain rule for derivatives and the change of variables formula for the Lebesgue integral. *The American Mathematical Monthly 76*(5) (1969) 514-520.

[24] G. Sobczyk, and O.L. Sánchez, Fundamental theorem of calculus. *Advances in Applied Clifford Algebras 21*(1) (2011) 221-231.

[25] C. Swartz, and B.S. Thomson, More on the fundamental theorem of calculus. *The American Mathematical Monthly 95*(7) (1988) 644-648.

[26] A.E. Taylor, L'Hospital's rule. *The American Mathematical Monthly 59*(1) (1952) 20-24.

[27] P.W. Thompson, Images of rate and operational understanding of the fundamental theorem of calculus. *Educational studies in mathematics 26*(2) (1994) 229-274.

[28] D.E. Varberg, Change of variables in multiple integrals. *The American Mathematical Monthly 78*(1) (1971) 42-45.

[29] W.H. Young, On integration with respect to a function of bounded variation. *Proceedings of the London Mathematical Society 2*(1) (1914) 109-150.