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Implementation of Newton-Raphson Iterative Method for Solving Non-Linear Equations in the Solow Economic Growth Model

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ABSTRACT

This study discusses the implementation of the Newton-Raphson iterative method in solving non-linear equations that appear in the Solow economic growth model. This model functions to analyze the influence of economic variables such as savin gs rates, capital depreciation, and output elasticity of capital on long-term economic equilibrium. The Newton-Raphson method is used because of its efficiency in achieving rapid convergence to a solution, although it is highly dependent on good initial guesses. Simulation results show that this method is able to find accurate numerical solutions with minimum number of iterations, so it has the potential to be applied in economic policy analysis that requires high precision.

***Keywords*:** Newton-Raphson, Non-Linear Equations, Solow Model, Economic Growth, Numerical Iteration.

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1. INTRODUCTION

The Solow economic growth model has long been an important analytical tool in understanding the dynamics of long-run economic growth. This model highlights the role of interactions between capital, labor, and technology in influencing output [1] [2]. In this context, the production function used often produces complex non-linear equations, reflecting the dynamic relationships between these factors [3].

However, solving a nonlinear equations in the Solow growth model is not a simple task. Analytical approaches are often inapplicable, so the use of numerical methods becomes a more practical option. One of the most well-known methods in this regard is the Newton-Raphson iterative method, which is known for its efficiency in finding the roots of nonlinear equations [4] [5]. Although this method has advantages in terms of convergence speed, challenges remain, especially related to stability and sensitivity to parameters [6].

The application of the Newton-Raphson method to the Solow growth model allows researchers to find steady-state solutions for capital and output, which has important implications for policy-making and economic planning [7]. On the other hand, the convergence challenges that often arise in the implementation of this method require special attention to ensure accurate and stable results [8].

Therefore, this study aims to implement the Newton-Raphson iterative method in solving the non-linear equation in the Solow economic growth model. This study is expected to provide more efficient and robust solutions in modern economic analysis.

2. MATERIAL AND METHODS

2.1. Economic growth

Economic growth is an increase in output per capita over a long period of time, usually lasting between 10 to 50 years or more. This growth has three main aspects: process, output per capita, and time period [9]. The process reflects how a country's economy develops over time, while output per capita reflects the total output divided by the population. The last aspect, namely the long term, states that economic growth is only achieved if the increase in output occurs over a sufficiently long period. Meanwhile, according to Samuelson, economic growth does not only include an increase in output per capita, but also an increase in real income and people's living standards [10].

The measurement of a country's economic growth is generally done through gross domestic product (GDP). There are three main methods in calculating GDP: the production approach, which calculates the value added of each production process to avoid double counting and produce GDP; the income approach, which combines all income from workers, employers, and capital owners to obtain National Income (NI); and the expenditure approach, which adds up all household and government expenditures to obtain Gross National Product (GNP) [11].

There are three key factors in the success of a country's economic growth: capital accumulation, population and labor growth, and technological progress [12]. Capital accumulation is obtained from part of the income that is saved or invested, including investment in physical assets and human resources. Population growth is related to the availability of labor that drives economic growth. Meanwhile, technological progress plays an important role in increasing productivity and driving long-term economic growth.

2.2. Non Linear Equations

Non-linear equations can be interpreted as equations that do not contain conditions like linear equations. Nonlinear equations are characterized by having exponents that differ from one. In addition, non-linear equations also include trigonometry, exponentials, and polynomials in general [13]. The polynomial equation is one of the most frequently encountered types of equations, generally represented in its standard form as shown in the corresponding equation (1).

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0 \quad (1)$$

With $a_0 \neq 0,$ for each $a_0, a_1, \ldots, a_n \in \mathbb{R}$

For the general form of a non-linear equation, the trigonometric form used can be seen in equations (2) and (3).

$$a \sin bx = 0 \quad (2)$$

$$a \cos bx = 0 \quad (3)$$

For each $.a, b \in \mathbb{R}$

Meanwhile, the general form of the non-linear equation, the exponential form, can be seen in equation (4).

$$a \,.\, e^{bx} = 0 \quad (4)$$

For each $.a, b \in \mathbb{R}$

2.3. Newton Raphson Method

Newton-Rhapson method is one method to calculate the roots of non-linear equations. Suppose $f(x)$ differentiable at $[a, b]$, then $f(x)$ has a tangent line at each point on $[a, b]$. The tangent line at $\big(x_0, f(x_0)\big)$ the point is an approximation of the graph $f(x)$ near the point $\big(x_0, f(x_0)\big)$. So the zero maker of the tangent line is an approximation of the zero maker $f(x)$.

To obtain the formula for the Newton-Raphson method, we begin by considering a function $f(x)$ and its corresponding graph. The method uses a tangent line approximation at a specific point on the curve, $\big(x_0, f(x_0)\big)$, to iteratively approach the root of the function. The gradient of the curve at this point is given by $f'(x_0)$, which serves as the slope of the tangent line. Using the equation for a straight line that passes through a point (x_0, y_0) with a gradient m, we can write the equation of the tangent line as:

$$y - y_0 = m(x - x_0) \quad (5)$$

Substituting $m = f'(x_0)$ and $y_0 = f(x_0)$, as the intercept at $(x_0, f(x_0))$, the equation of the tangent line becomes:

$$y - f(x_0) = f'(x_0)(x - x_0) \quad (6)$$

To determine the point where this tangent line intersects the axis$-x$, we set $y = 0$ giving:

$$(7) \quad 0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$-f(x_0) = x_1 f'(x_0) - x_0 f'(x_0)$$

$$x_1 f'(x_0) = x_0 f'(x_0) - f(x_0)$$

$$x_1 = x_0 \frac{f'(x_0)}{f'(x_0)} - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Written in general form it becomes:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (8)$$

2.4. Solow's Theory

The Solow growth theory, which is an extension of the Harrod-Domar theory, introduces the possibility of variations in interest rates and wages. Unlike the Harrod-Domar model which assumes a sticky interest rate and a constant wage rate in the long run, the Solow model introduces the concept of economic growth that depends on capital and labor as the main inputs [2] [14]. Its production function is written as:

(9) Y=f(K,L)

where Y is output, K is capital, and L is labor. By taking technology into account as a factor that increases the efficiency of labor and capital, the production function can be expanded to:

(10) Y=f[(K,L)·E]

Here, E represents the labor efficiency gained from technology. Technology can be knowledge and skills that facilitate the production process, and increased labor efficiency can occur with improvements in education, health, and community skills. In capital efficiency, technology involves the use of machines or tools that speed up the production process.

In the Solow model, technology is considered an exogenous factor, meaning that technological progress is explained externally and is referred to as total factor productivity. Therefore, the Solow model is often referred to as an exogenous growth model. To improve this limitation, endogenous growth models were developed by including additional variables that can explain the contribution of technology to long-term economic growth [14].

The basic model of Solow economic growth with the Cobb-Douglas production function is defined as follows:

$$(11) \qquad Y = A.K^{a}.L^{1-a}$$

With

Y: output or GDP

A: productivity factor

K: Capital

L: workforce

a: parameter of output elasticity with respect to capital

The above model also involves the capital accumulation function:

$$(12) \qquad \Delta K = sY - \delta K$$

Where,
s: savings rate
δ: capital depreciation rate

Steady State Equation
Steady state is reached when capital accumulation (stops changing i.e.:ΔK)

$$\Delta K = 0,$$

So that

$$sY = \delta K$$

The value (steady state capital) can be found using the following equation: K^*

$$(13) \qquad sA.(K^*)^{a}.L^{1-a} = \delta K^*$$

Steady state (is a condition where capital per capita is stable because new investment is only sufficient to replace capital depreciation.K^*)
So we get the following non-linear equation:

$$(14) \quad f(K^*) = sA.(K^*)^a.L^{1-a} - \delta K^* = 0$$

The equation $f(K^*)$ does not have a simple analytical solution, so the Newton Raphson method is needed to find the value K^*.

The Newton Raphson method is applied to the above Solow economic growth model. The Newton Raphson iteration formula is written as follows:

$$(15) \quad K_{n+1} = K_n - \frac{f(K_n)}{f'(K_n)}$$

With $f(K^*)$ that already defined, the first derivative $f'(K)$ can be written as follows:

$$(16) \quad f'(K) = sA.a.K^{a-1}.L^{1-a} - \delta$$

2.5. Data

Table 1 provides a detailed overview of the essential economic parameters employed in the Solow Growth Model analysis for Thailand in 2023. These parameters encompass the national savings rate, the labor force population, the rate of capital depreciation, the elasticity of output with respect to capital, and the total factor productivity. Each parameter is derived from credible sources, including international organizations and renowned academic studies, ensuring the robustness and validity of the model for understanding Thailand's long-term economic growth trajectory**.**

Table 1. Solow Economic Model Parameter for Thailand (2023).

Parameter	Value	Source
National Savings Rate (s)	0.25	The World Bank. (n.d.). *Gross Savings (% of GDP) - Thailand.* Retrieved from World Bank[15].
Labor Force Population (L)	40 million	International Labour Organization (ILO). (n.d.). *ILOSTAT Database - Thailand Labour Force.* Retrieved from ILOSTAT[16].

Capital Depreciation Rate (δ)	0.05	Based on estimates from Caselli, F. (2005). *Accounting for Cross-Country Income Differences*. In *Handbook of Economic Growth* (Vol. 1, Elsevier)[17].
Elasticity of Output to Capital (α)	0.3	Barro, R. J., & Sala-i-Martin, X. (2004). *Economic Growth (2nd ed.)*. MIT Press[18].
Total Factor Productivity (TFP) (A)	1	Organisation for Economic Co-operation and Development (OECD). (n.d.). *Productivity Statistics*. Retrieved from OECD Statistics[19].

3. RESULT AND DISCUSSION

In this study, the Python programming language was used for data processing, numerical analysis, and visualization to solve and interpret the Solow economic growth model. The Newton-Raphson iterative method was applied through a specially designed Python script with a convergence tolerance of 10^{-6}. This approach ensures accurate and efficient calculations.

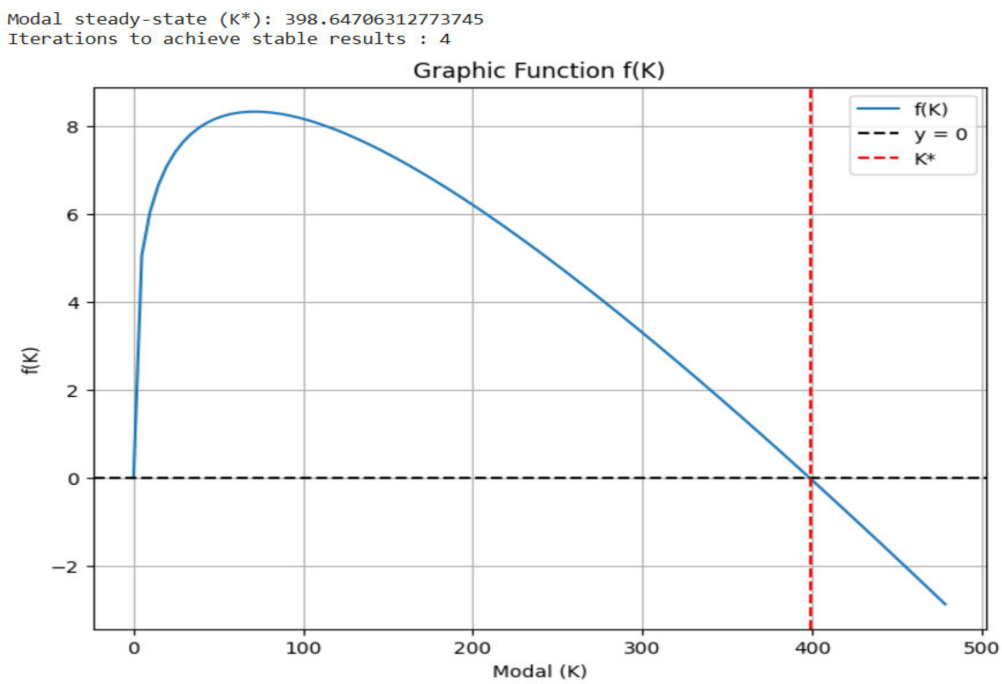

Figure 1. Graphic function of f(K) and calculation result.

The graph displayed in the image represents the function f(K) from the Solow economic growth model, which shows the relationship between capital (K) and the output produced in the economy. The blue curve represents the function f(K), which increases initially as capital rises, reaches a peak, and then begins to decline. The black dashed line represents the horizontal axis where the output f(K) equals zero, marking the threshold where the economy transitions from positive to negative output. The red dashed vertical line indicates the steady-state capital (K*), which is the optimal amount of capital for stable long-term economic growth. At this point, net investment is zero, meaning the economy is in equilibrium where the level of investment is sufficient to offset capital depreciation. The graph visually aids in understanding how the economy converges to a steady-state capital level over time, as shown by the intersection of the curve with the axis and the steady-state point.

The steady-state capital value or long-term equilibrium capital is found at 398.64. This means that in the Solow model in Thailand, the optimal capital level for long-term economic growth is around 398.64. This value reflects the level of capital in the economy is in a stable condition, the investment (savings) made is sufficient to replace the capital that is shrinking due to depreciation. This means that at this point, economic growth becomes constant, without a continuous increase or decrease in capital per capita.

Achieving stable results required four iterations, with a predefined error tolerance of 10^{-6}, demonstrating the efficiency of this method in locating the roots of non-linear equations. The Newton-Raphson iterative method is applied to the non-linear function of capital, which in this model, is related to the relationship between capital (K) and the output produced in the economy. This method updates the capital value at each iteration until it approaches the steady-state point with high precision.

4. CONCLUSIONS

The capital of 398.64 represents the ideal amount of capital for the economy to be in a stable growth condition. If the capital is lower than this value, the economy is at risk of declining output due to underinvestment. Conversely, if it is too high, excess capital may not be offset by a commensurate increase in productivity. The calculation results show that the Newton-Raphson iterative method can be effectively implemented to solve the non-linear equations in the Solow economic growth model. With relatively few iterations, this method successfully achieves accurate results, reflecting the steady-state capital that is essential for balanced economic growth in the long run.

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