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Solutions of the Difference Equations by the New Core-Shell Approach

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ABSTRACT

The recently proposed core-shell approach for solving differential equations is applied to find solutions of difference equations for the first time. The new method provides an alternative to the existing methods in search of analytical solutions of difference equations. The method is applied mainly to homogenous and non-homogenous variable coefficient second order linear difference equations. Some theorems on the general form of the solutions are given for such equations. Finally, the Cauchy-Euler difference equation is solved by the method. For higher order variable coefficient difference equations, solutions can be constructed in an analogous way. For first and second order equations, the general solutions given in theorems can directly be implemented to find the solutions for the specific equations. The analysis presented will not only provide an alternative solution method, but will also add insight to the understanding of difference equations and the nature of solutions.

Keywords: Difference Equations; Solution Algorithms; Decomposition, Core-Shell Approach

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1. INTRODUCTION

Similar to differential equations, difference equations proved to be very effective in modelling real world problems. While differential equations represent continuous variations in the physical quantities, the difference equations involve problems where changes occur at finite discrete stations. A beam can be modelled as a continuous structural element where the mass is distributed over the length continuously, which leads to a differential equation in modelling the problem. Alternatively, the beam can be assumed to be composed of several concentrated masses connected to each other, which in turn require modeling in the form of difference equations. Furthermore, some solution methods of differential equations, such as the series solutions or the Frobenius method, lead to difference equations as a result [11]. The famous equation producing Fibonacci sequence, which is a pattern encountered frequently in nature, is a constant coefficient second order difference equation [12]. The differential equations and difference equations have close ties between them, the former can be considered as the continuous form and the latter the discrete form. To identify the analogy, the difference equation has to be written in operator form [11]. The continuous form of the Fibonacci difference equation, which is named as the Fibonacci Differential equation, is derived using the analogy between them [12].

Similar to the case of differential equations, exact analytical solutions are rare in the case of variable coefficient linear difference equations as well as nonlinear equations. For constant coefficient difference equations, the analytical solutions are well established [11] although there are still researchers working on the topic. First and second order linear difference equations with constant coefficients and their solutions were investigated from the pedagogical point of view with a minimum theoretical background [6]. A new approach has been presented for non-homogenous second order linear difference equations having constant coefficients [14]. For variable coefficients, various methods were developed to solve second and higher order linear difference equations. Usually, the coefficients have an assumed special form and solutions were constructed under these restrictions. The non-homogeneous second-order linear difference equation with periodic coefficients was transferred to an equivalent family of non-homogeneous linear difference equations with constant coefficients, and new solutions were proposed [16]. Using Green function, explicit solutions of a general second order linear difference equation were presented [4]. For higher order linear difference equations, several solution methods were proposed. Lie Group theory approach was applied to difference equations for investigating the periodicity and asymptotic behaviour of the solutions [5]. Based on the hybrid Legendre and Taylor polynomials, higher order difference equations were solved numerically [8]. Linear difference equations with variable coefficients under mixed conditions were solved by Taylor polynomial approximation [7]. Asymptotic representation of solutions of a system of linear difference equations with special variable coefficients was investigated [3]. A solvable third-order nonlinear difference equation system with variable real coefficients was treated [15]. Reduction of system of linear difference equations to a system with constant coefficients were investigated for Lyapunov type substitutions [2]. The explicit solution of a special linear difference equation of unbounded order with variable coefficients is presented [10]. Regarding the nonlinear difference equations, well established methods and theorems are non-existent, and special methods were used for constructing analytical and semi-analytical solutions. A closed form solution was presented for a nonlinear system of difference equations with non-zero real coefficients [9]. Special third-order nonlinear difference equation systems were solved, and it was shown that some of the solutions were periodic [1]. Periodicity of solutions was outlined for a specific nonlinear differential equation system [17].

In this paper, the core-shell approach developed for differential equations [13] very recently is applied to difference equations for the first time. To achieve this, the difference equation is expressed in a difference operator form first and decomposed into lower order operators. Then the original equation, called the embedded equation, is expressed as separate equations in which one of the equations is called core equation and the others shell equations. For second order difference equations, there is only one shell equation, but for higher order equations, there may be several shell equations named the outer shell and the intermediate shells. First, the outer shell equation is solved, then the intermediate shells if exist, and then finally the core equation is solved to reach the solution of the original embedded equation.

The study is confined to homogeneous and non-homogeneous second order variable coefficient linear difference equations, although the method can be applied to higher order linear equations as well as to nonlinear equations. Solutions for higher order linear equations as well as nonlinear equations are left for further studies. Solution algorithms for homogeneous and non-homogeneous second order variable coefficient linear difference equations are constructed and summarized in the form of theorems. The analytical solution can be found if the decomposition into core and shell is successful for a given specific equation. As with the other methods, if the difference equation does not possess an exact analytical solution, then the method will fail, and decomposition into core and shell parts will be impossible. In such cases, employment of semi-analytical or numerical techniques is inevitable. The main contribution in this paper is to develop an alternative and systematic way of producing solutions for second order variable coefficient linear difference equations. Many worked examples are treated, including the Cauchy-Euler difference equations.

2. PRELIMINARIES

The discrete first and second derivatives are defined as:

$$(1) \quad Dy_k = y_{k+1} - y_k \ ,$$

$$(2) \quad D^2 y_k = D(Dy_k) = D(y_{k+1} - y_k) = y_{k+2} - 2y_{k+1} + y_k \ .$$

The operator multiplication is somewhat different from the differentiation of two functions multiplied with each other:

$$(3) \quad D(u_k v_k) = u_{k+1} v_{k+1} - u_k v_k \ .$$

For the first order variable coefficient non-homogeneous linear difference equation:

$$(4) \quad y_{k+1} = a_k y_k + b_k \ , \quad k = 0,1,2,\dots ,$$

the general solution is [11]

$$(5) \quad y_k = \prod_{j=0}^{k-1} a_j \left(A_0 + \sum_{j=0}^{k-1} \frac{b_j}{\prod_{s=0}^{j} a_s} \right) \ , \ \ k = 1,2,\dots \ ,$$

where the constant A_0 is determined from the initial condition.

2.1. Core-Shell Approach for Linear Homogenous Second Order Equations

Consider the variable coefficient linear homogeneous second order difference equation

$$(6) \quad y_{k+2} + p_k y_{k+1} + q_k y_k = 0\,, \quad k = 0,1,2,\ldots \quad .$$

One may write the second order equation (6) in operator form decomposed into first order discrete operators

$$(7) \quad (D + a_k + 1)(D + b_k + 1)y_k = 0\,.$$

a_k and b_k has to be defined in terms of the coefficients p_k and q_k. Applying the inner operator using definition (1)

$$(8) \quad (D + b_k + 1)y_k = y_{k+1} + b_k y_k \;.$$

Then the first operator is applied

$$(9) \quad (D + a_k + 1)(y_{k+1} + b_k y_k) = y_{k+2} + (a_k + b_{k+1})y_{k+1} + a_k b_k y_k = 0\,.$$

Comparing (9) and (6), one may conclude

$$(10) \quad p_k = a_k + b_{k+1}$$

$$(11) \quad q_k = a_k b_k\,.$$

Solving a_k from (10)

$$(12) \quad a_k = p_k - b_{k+1}\,,$$

and substituting into (11)

$$(13) \quad q_k = (p_k - b_{k+1})b_k\,,$$

or

$$(14) \quad b_{k+1}b_k - p_k b_k + q_k = 0\;.$$

Therefore, is solved from (14) and a_k is found from (12). The equivalent of (14) in differential equation version of the core-shell approach is the well-known Riccati equation [13]. The decomposition is possible if (14) is analytically solvable. To apply the core-shell approach, one has to identify the core and shell parts of the original embedded difference equation (6).

$$(15) \quad (D + b_k + 1)y_k = u_k \quad \to \quad y_{k+1} + b_k y_k = u_k \quad \text{(Core Equation).}$$

$$(16) \quad (D + a_k + 1)u_k = 0 \quad \to \quad u_{k+1} + a_k u_k = 0 \quad \text{(Shell Equation).}$$

First the shell equation has to be solved.

$$(17) \quad u_1 = -a_0 u_0$$

$$(18) \quad u_2 = -a_1 u_1 = (-a_0)(-a_1)u_0$$

$$(19) \quad u_3 = -a_2 u_2 = (-a_0)(-a_1)(-a_2)u_0 \, .$$

By induction

$$(20) \quad u_k = c_2 \prod_{j=0}^{k-1}(-a_j) \ , \ \ k = 1,2, \dots \ ,$$

where $c_2 = u_0$. Substituting for u_k into the core equation

$$(21) \quad y_{k+1} + b_k y_k = c_2 \prod_{j=0}^{k-1}(-a_j) \, ,$$

and multiplying the equation by $1/\prod_{j=0}^{k}(-b_j)$,

$$(22) \quad \frac{y_{k+1}}{\prod_{j=0}^{k}(-b_j)} - \frac{y_k}{\prod_{j=0}^{k-1}(-b_j)} = c_2 \frac{\prod_{j=0}^{k-1}(-a_j)}{\prod_{j=0}^{k}(-b_j)} \quad .$$

If one defines

$$(23) \quad A_k = \frac{y_k}{\prod_{j=0}^{k-1}(-b_j)} \quad ,$$

the equation is with change of the indices

$$(24) \quad A_{j+1} - A_j = c_2 \frac{\prod_{m=0}^{j-1}(-a_m)}{\prod_{m=0}^{j}(-b_m)} \, .$$

Summing both sides, $j = 0,1,..,k-1$, the intermediate A_j terms cancel

$$(25) \quad A_k - A_0 = c_2 \sum_{j=0}^{k-1} \frac{\prod_{m=0}^{j-1}(-a_m)}{\prod_{m=0}^{j}(-b_m)} \, .$$

Substituting (23) into (25) and solving for y_k

$$(26) \quad y_k = \prod_{j=0}^{k-1}(-b_j) \left\{ c_1 + c_2 \sum_{j=0}^{k-1} \frac{\prod_{m=0}^{j-1}(-a_m)}{\prod_{m=0}^{j}(-b_m)} \right\} ,$$

where $A_0 = c_1$. Results are summarized in the following theorem.

Theorem 1.

For the variable coefficient homogenous second order linear difference equation

$$(27)\quad y_{k+2} + p_k y_{k+1} + q_k y_k = 0\,, \quad k = 0,1,2,\dots\,,$$

if the equation can be written in the decomposed discrete operator form

$$(28)\quad (D + a_k + 1)(D + b_k + 1)y_k = 0\,,$$

then the general solution immediately follows from the core-shell approach

$$(29)\quad y_k = \prod_{j=0}^{k-1}(-b_j)\left\{c_1 + c_2 \sum_{j=0}^{k-1} \frac{\prod_{m=0}^{j-1}(-a_m)}{\prod_{m=0}^{j}(-b_m)}\right\},$$

where a_k and b_k satisfy

$$(30)\quad a_k = p_k - b_{k+1}$$

$$(31)\quad b_{k+1}b_k - p_k b_k + q_k = 0\,,$$

for $b_k \neq 0$.

Proof 1.

See the above calculations

Remark 1.

If $p_k = p$ and $q_k = q$ are constants, then $a_k = a$ and $b_k = b$ always possess constant solutions. From (31)

$$(32)\quad b^2 - pb + q = 0\,,$$

which is a quadratic equation. Without loss of generality, one may take

$$(33)\quad b = \frac{1}{2}\left(p - \sqrt{p^2 - 4q}\right),$$

and from (30)

$$(34)\quad a = \frac{1}{2}\left(p + \sqrt{p^2 - 4q}\right)\,.$$

The final solution is found upon substitution into (29)

Several sample problems are solved using the general algorithm given in Theorem 1.

***Example* 1**

Consider the constant coefficient second order equation

$$(35) \quad y_{k+2} - 4y_k = 0\,, \quad k = 0,1,2,\dots\ .$$

From the equation $p = 0$ and $q = -4$. From (33) and (34), $a = 2$ and $b = -2$. From (29)

$$(36) \quad y_k = \prod_{j=0}^{k-1}(2)\left\{c_1 + c_2 \sum_{j=0}^{k-1} \frac{\prod_{m=0}^{j-1}(-2)}{\prod_{m=0}^{j}(2)}\right\}$$

$$(37) \quad y_k = 2^k \left\{c_1 + c_2 \sum_{j=0}^{k-1} \frac{(-2)^j}{(2)^{j+1}}\right\}$$

$$(38) \quad y_k = 2^k \left\{c_1 + \frac{c_2}{2} \sum_{j=0}^{k-1}(-1)^j\right\}$$

$$(39) \quad y_k = c_1 2^k + c_2 2^{k-1}(-1)^{k-1} \quad .$$

Redefining new c_2, the solution is

$$(40) \quad y_k = c_1 2^k + c_2(-2)^k \quad .$$

***Example* 2**

Consider the constant coefficient second order equation

$$(41) \quad y_{k+2} - 6y_{k+1} + 9y_k = 0\,, \quad k = 0,1,2,\dots \quad .$$

From the equation $p = -6$ and $q = 9$. From (33) and (34), $a = -3$ and $b = -3$. From (29)

$$(42) \quad y_k = \prod_{j=0}^{k-1}(3)\left\{c_1 + c_2 \sum_{j=0}^{k-1} \frac{\prod_{m=0}^{j-1}(3)}{\prod_{m=0}^{j}(3)}\right\} \quad ,$$

$$(43) \quad y_k = 3^k \left\{c_1 + c_2 \sum_{j=0}^{k-1} \frac{(3)^j}{(3)^{j+1}}\right\},$$

$$(44) \quad y_k = 3^k \left\{c_1 + \frac{c_2}{3} k\right\},$$

Redefining new c_2, the solution is

$$(45) \quad y_k = c_1 3^k + c_2 k 3^k \quad .$$

***Example* 3**

Consider the constant coefficient second order equation

$$(46) \quad y_{k+2} + y_k = 0\,, \quad k = 0,1,2, \dots \ .$$

From the equation $p = 0$ and $q = 1$. From (33) and (34), $a = i$ and $b = -i$. From (29)

$$(47) \quad y_k = \prod_{j=0}^{k-1}(i)\left\{c_1 + c_2 \sum_{j=0}^{k-1} \frac{\prod_{m=0}^{j-1}(-i)}{\prod_{m=0}^{j}(i)}\right\},$$

which is finally after redefining c_2,

$$(48) \; y_k = c_1 i^k + c_2(-i)^k \ .$$

Since $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, defining new constants, the solution is

$$(49) \quad y_k = c_1 \cos\frac{k\pi}{2} + c_2 \sin\frac{k\pi}{2} \ .$$

The following examples are variable coefficient equations

***Example* 4**

Consider the variable coefficient second order equation

$$(50) \quad y_{k+2} - (k+1)y_{k+1} + k y_k = 0\,, \quad k = 0,1,2, \dots \ .$$

From the equation $p_k = -(k+1)$ and $q_k = k$. From (31)

$$(51) \quad b_{k+1} b_k + (k+1) b_k + k = 0$$

whose solution is $b_k = -1$. From 30, $a_k = -k$. Substituting into (29) and starting the initial values from nonzero values

$$(52) \quad y_k = \prod_{j=1}^{k-1}(1)\left\{c_1 + c_2 \sum_{j=2}^{k-1} \frac{\prod_{m=1}^{j-1}(m)}{\prod_{m=1}^{j}(1)}\right\} ,$$

which yields

$$(53) \quad y_k = c_1 + c_2 \sum_{j=2}^{k-1}(j-1)! \ .$$

The solution can be verified by substituting into the original difference equation.

***Example* 5**

Consider the variable coefficient second order equation

$$(54)\quad y_{k+2} - (2k+1)y_{k+1} + k^2 y_k = 0\,, \quad k = 0,1,2,\dots\,,$$

From the equation $p_k = -(2k+1)$ and $q_k = k^2$. From (31)

$$(55)\quad b_{k+1}b_k + (2k+1)b_k + k^2 = 0\,,$$

whose solution is $b_k = -k$. From 30, $a_k = -k$. Substituting into (29) and starting the initial values from nonzero values

$$(56)\quad y_k = \prod_{j=1}^{k-1}(j)\left\{c_1 + c_2\sum_{j=1}^{k-1}\frac{\prod_{m=1}^{j-1}(m)}{\prod_{m=1}^{j}(m)}\right\},$$

which yields

$$(57)\quad y_k = (k-1)!\left(c_1 + c_2\sum_{j=1}^{k-1}\frac{1}{j}\right)\,.$$

The solution can be verified by substituting into the original difference equation.

***Example* 6**

Consider the variable coefficient second order equation

$$(58)\quad y_{k+2} - \frac{(2k+1)}{k(k+1)}y_{k+1} + \frac{1}{k^2}y_k = 0\,, \quad k = 1,2,\dots\,.$$

From the equation $p_k = -\frac{(2k+1)}{k(k+1)}$ and $q_k = \frac{1}{k^2}$. From (31)

$$(59)\quad b_{k+1}b_k + \frac{(2k+1)}{k(k+1)}b_k + \frac{1}{k^2} = 0\,,$$

whose solution is $b_k = -\frac{1}{k}$. From 30, $a_k = -\frac{1}{k}$. Substituting into (29) and starting the initial values from nonzero values

$$(60)\quad y_k = \prod_{j=1}^{k-1}\left(\frac{1}{j}\right)\left\{c_1 + c_2\sum_{j=1}^{k-1}\frac{\prod_{m=1}^{j-1}\left(\frac{1}{m}\right)}{\prod_{m=1}^{j}\left(\frac{1}{m}\right)}\right\}\,,$$

which yields after redefining the constant c_2

$$(61)\quad y_k = c_1\frac{1}{(k-1)!} + c_2\frac{k}{(k-2)!}\,.$$

The solution can be verified by substituting into the original difference equation.

2.2. Core-Shell Approach for Linear Non-Homogenous Second Order Equations

Consider the variable coefficient linear non-homogeneous second order difference equation

$$(62)\quad y_{k+2} + p_k y_{k+1} + q_k y_k = f_k\,, \quad k = 0,1,2,\ldots,$$

which can be written in decomposed operator form

$$(63)\quad (D + a_k + 1)(D + b_k + 1)f_k = 0\,,$$

where

$$(64)\quad p_k = a_k + b_{k+1}$$

$$(65)\quad q_k = a_k b_k.$$

Eliminating a_k from above

$$(66)\quad b_{k+1}b_k - p_k b_k + q_k = 0\,.$$

Solving then a_k from (64)

$$(67)\quad a_k = p_k - b_{k+1}.$$

The decomposition is possible if (66) is solvable. To apply the core-shell approach, similar to the previous section, one has to identify the core and shell parts of the original embedded difference equation (62)

$$(68)\quad (D + b_k + 1)y_k = u_k \quad \rightarrow \quad y_{k+1} + b_k y_k = u_k \quad \text{(Core Equation)}\,,$$

$$(69)\quad (D + a_k + 1)u_k = f_k \quad \rightarrow \quad u_{k+1} + a_k u_k = f_k \quad \text{(Shell Equation)}\,.$$

First the shell equation has to be solved. Multiply both sides of (69) by $1/\prod_{j=0}^{k}(-a_j)$,

$$(70)\quad \frac{u_{k+1}}{\prod_{j=0}^{k}(-a_j)} - \frac{u_k}{\prod_{j=0}^{k-1}(-a_j)} = \frac{f_k}{\prod_{j=0}^{k}(-a_j)}\quad.$$

If one defines

$$(71)\quad A_k = \frac{u_k}{\prod_{j=0}^{k-1}(-a_j)}\,,$$

the equation is with change of the indices

$$(72)\quad A_{j+1} - A_j = \frac{f_j}{\prod_{m=0}^{j}(-a_m)}\,.$$

Summing both sides $j = 0,1,..,k-1$, the intermediate A_j terms cancel

$$(73)\quad A_k - A_0 = \sum_{j=0}^{k-1} \frac{f_j}{\prod_{m=0}^{j}(-a_m)} \ ,$$

where $A_0 = 0$ for a particular solution. Substituting (71) into (73) and solving for u_k

$$(74)\quad u_k = \prod_{j=0}^{k-1}(-a_j) \sum_{j=0}^{k-1} \frac{f_j}{\prod_{m=0}^{j}(-a_m)} \ .$$

The core equation (68) is solved next. Multiplying both sides of the equation by $1/\prod_{j=0}^{k}(-b_j)$,

$$(75)\quad \frac{y_{k+1}}{\prod_{j=0}^{k}(-b_j)} - \frac{y_k}{\prod_{j=0}^{k-1}(-b_j)} = \frac{u_k}{\prod_{j=0}^{k}(-b_j)} \ .$$

Defining

$$(76)\quad B_k = \frac{y_k}{\prod_{j=0}^{k-1}(-b_j)} \ ,$$

the equation is with change of the indices

$$(77)\quad B_{j+1} - B_j = \frac{u_j}{\prod_{m=0}^{j}(-b_m)} \ .$$

Summing both sides $j = 0,1,..,k-1$, the intermediate B_j terms cancel

$$(78)\quad B_k - B_0 = \sum_{j=0}^{k-1} \frac{u_j}{\prod_{m=0}^{j}(-b_m)} \ ,$$

where $B_0 = 0$ for a particular solution. From (76), the particular solution is

$$(79)\quad (y_k)_p = \prod_{j=0}^{k-1}(-b_j) \sum_{j=0}^{k-1} \left(\frac{\prod_{m=0}^{j-1}(-a_m)}{\prod_{m=0}^{j}(-b_m)} \sum_{m=0}^{j-1} \frac{f_m}{\prod_{n=0}^{m}(-a_n)} \right) .$$

Results are summarized in the following theorem.

Theorem 2.

For the variable coefficient non-homogenous second order linear difference equation

$$(80)\quad y_{k+2} + p_k y_{k+1} + q_k y_k = f_k \ , \quad k = 0,1,2, \dots \ ,$$

if the equation can be written in the decomposed discrete operator form

$$(81)\quad (D + a_k + 1)(D + b_k + 1) y_k = f_k \ ,$$

then the particular solution immediately follows from the core-shell approach

$$(82)\quad (y_k)_p = \prod_{j=0}^{k-1}(-b_j)\sum_{j=0}^{k-1}\left(\frac{\prod_{m=0}^{j-1}(-a_m)}{\prod_{m=0}^{j}(-b_m)}\sum_{m=0}^{j-1}\frac{f_m}{\prod_{n=0}^{m}(-a_n)}\right),$$

where a_k and b_k satisfy

$$(83)\quad a_k = p_k - b_{k+1}\,,$$

$$(84)\quad b_{k+1}b_k - p_k b_k + q_k = 0\,,$$

for $b_k \neq 0$.

Proof 2.

See the above calculations

***Example* 7**

Find the particular solution of

$$(85)\quad y_{k+2} - 2y_{k+1} + y_k = \cos k\,,\quad k = 0,1,2,\ldots\,.$$

From the equation $p_k = -2$ and $q_k = 1$. From (84)

$$(86)\quad b_{k+1}b_k + 2b_k + 1 = 0\,,$$

whose solution is $b_k = -1$. From 83, $a_k = -1$. Substituting into (82)

$$(87)\quad (y_k)_p = \prod_{j=0}^{k-1}(1)\sum_{j=0}^{k-1}\left(\frac{\prod_{m=0}^{j-1}(1)}{\prod_{m=0}^{j}(1)}\sum_{m=0}^{j-1}\frac{\cos m}{\prod_{n=0}^{m}(1)}\right),$$

which yields

$$(88)\quad (y_k)_p = \sum_{j=1}^{k-1}\left(\sum_{m=0}^{j-1}\cos m\right).$$

The particular solution can be verified by substituting into the original difference equation.

***Example* 8**

Find the particular solution of

$$(89)\quad y_{k+2} - (2k+3)y_{k+1} + (k+1)^2 y_k = k(k+1)\,,\quad k = 1,2,\ldots\,.$$

From the equation $p_k = -(2k+3)$ and $q_k = (k+1)^2$. From (84)

$$(90)\quad b_{k+1}b_k + (2k+3)b_k + (k+1)^2 = 0\,,$$

whose solution is $b_k = -(k+1)$. From 83, $a_k = -(k+1)$. Substituting into (82)

$$(91) \quad (y_k)_p = \prod_{j=0}^{k-1}(j+1)\sum_{j=0}^{k-1}\left(\frac{\prod_{m=0}^{j-1}(m+1)}{\prod_{m=0}^{j}(m+1)}\sum_{m=0}^{j-1}\frac{m(m+1)}{\prod_{n=0}^{m}(n+1)}\right) \quad ,$$

which yields

$$(92) \quad (y_k)_p = k!\sum_{j=2}^{k-1}\left(\frac{1}{j+1}\sum_{m=1}^{j-1}\frac{1}{(m-1)!}\right) \; .$$

The solution can be verified by substituting into the original difference equation.

Note that particular solutions are valid if none of the intermediate coefficients are zero. Assume that for the first order difference equation

$$(93) \quad y_{k+1} + a_k y_k = f_k, \qquad k = 0,1,2,\dots \; ,$$

$a_p = 0$ for some specific value of $k = p$. Then it can be shown that, the solution constitutes of three parts

$$(94) \quad y_k = \prod_{j=0}^{k-1}(-a_j)\left(y_0 + \sum_{j=0}^{k-1}\left(\frac{f_j}{\prod_{m=0}^{j}(-a_m)}\right)\right), \quad k = 0,1,2,\dots p \; ,$$

$$(95) \quad y_{p+1} = f_p \; ,$$

$$(96) \quad y_k = \prod_{j=p+1}^{k-1}(-a_j)\left(f_p + \sum_{j=p+1}^{k-1}\left(\frac{f_j}{\prod_{m=p+1}^{j}(-a_m)}\right)\right), \quad k = p+2, p+3, \dots$$

2.3. The Cauchy-Euler Difference Equation

One of the famous variable coefficient second order difference equations which possess exact analytical solutions is the Cauchy-Euler Difference equation. In discrete operator form, the equation is [11],

$$(97) \quad k(k+1)D^2 y_k + ckDy_k + dy_k = 0 \; ,$$

for some constants c and d. Using (1) and (2) and dividing the equation by $k(k+1)$ the equation becomes

$$(98) \quad y_{k+2} + \left(\frac{c}{k+1} - 2\right)y_{k+1} + \left(1 - \frac{c}{k+1} + \frac{d}{k(k+1)}\right)y_k = 0 \; .$$

Comparing with the standard form given in (27)

$$(99) \quad p_k = \frac{c}{k+1} - 2, \quad q_k = 1 - \frac{c}{k+1} + \frac{d}{k(k+1)} \qquad .$$

Substituting into (10) and (11)

$$(100)\quad a_k + b_{k+1} = \frac{c}{k+1} - 2\,,$$

$$(101)\quad a_k b_k = 1 - \frac{c}{k+1} + \frac{d}{k(k+1)}\,.$$

For the above equations, assume solutions of the form

$$(102)\quad a_k = -1 + \frac{a}{k+1}\,,\qquad b_k = -1 + \frac{b}{k}\,.$$

After some algebra, the constants are found to be

$$(103)\quad a = \frac{1}{2}\left(c + 1 - \sqrt{(1-c)^2 - 4d}\right),$$

$$(104)\quad b = \frac{1}{2}\left(c - 1 + \sqrt{(1-c)^2 - 4d}\right).$$

Since a_k and b_k are known, the result immediately follows from Theorem 1.

***Example* 9**

Consider the Cauchy-Euler difference equation

$$(105)\quad k(k+1)D^2 y_k - 5kDy_k + 8y_k = 0\,.$$

Since $c = -5$ and $d = 8$, from (103) and (104), $a = -3$ and $b = -2$. Then from (102)

$$(106)\quad a_k = -1 - \frac{3}{k+1}\,,\qquad b_k = -1 - \frac{2}{k}\,.$$

Substituting (106) into (29)

$$(107)\quad y_k = \prod_{j=1}^{k-1}\left(1 + \frac{2}{j}\right)\left\{c_1 + c_2 \sum_{j=0}^{k-1} \frac{\prod_{m=0}^{j-1}\left(1 + \frac{3}{m+1}\right)}{\prod_{m=1}^{j}\left(1 + -\frac{2}{m}\right)}\right\},$$

$$(108)\quad y_k = \frac{k(k+1)}{2}\left\{c_1 + c_2 \frac{1}{3} \sum_{j=0}^{k-1} (j+3)\right\}\,.$$

Evaluating the sum and redefining the constants, the final solution is

$$(109)\quad y_k = c_1 k(k+1) + c_2 k(k+1)(k+2)(k+3)\,,$$

which is the same solution obtained by other methods [11].

3. CONCLUDING REMARKS

This paper develops a new systematic way of solving difference equations. Although the variable coefficient linear difference equations are treated, the method can be applied to higher order equations as well as nonlinear equations. The method is based on decomposing the difference equation into its core and shell components and solving the decomposed equations starting from the shell equations and ending up with the core equation. The general solution algorithms for homogeneous and non-homogeneous variable coefficient second order linear equations are given in the form of theorems. The method produces exact analytical solutions subject to the condition that the equation is decomposable. For nonlinear equations, the decomposition process is not straightforward as in the case of linear equations. Therefore some intuition and ad hoc methods are needed to see the shell and core parts. The solution of nonlinear equations is left for further studies. The algorithms can also be used as reverse problems: Instead of finding a solution to a given difference equation, from a given solution, the equations that possess exact analytical solutions can also be derived. This feature may be useful for textbook authors and instructors to develop difference equations with exact analytical solutions.

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