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Limitations of Multiplicative Functional Analysis

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ABSTRACT

The usual modulus function of a real number is the maximum of the real number and the additive inverse of the real number. The multiplicative modulus function of a positive real number is the maximum of the positive real number and the multiplicative inverse of the positive real number. This is the beginning of the "multiplicative" mathematics. The possibilities for extensions of various branches of mathematics have already been studied for "multiplicative" branches. This article presents a study on possibilities for extensions of functional analysis to multiplicative functional analysis.

***Keywords*:** Multiplicative modulus function, Topological vector spaces, Transformations, Measures.

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1. INTRODUCTION

The set $\mathbb{R}$ of all real numbers is a field with respect to the usual addition and the usual multiplication. In particular, this set contains the additive inverse of every real number. The usual modulus function on $\mathbb{R}$ is defined by $|x| = \max\{x, -x\}, \forall x \in \mathbb{R}$. Here $-x$ is the additive inverse of x. What would happen if the multiplicative inverse is considered instead of the additive inverse? The number 0 has no multiplicative inverse. So, there is a need to restrict the set of all real numbers, when multiplicative inverses are considered. Let us restrict only to the open ray $(0, \infty) = \{x \in \mathbb{R}: x > 0\}$. The set of all positive numbers is considered just to get the multiplicative inverses so that the set becomes a commutative group under multiplication. Let us now define the multiplicative modulus function [15, 16, 18] on $(0, \infty)$ by $|x|_\times = \max\left\{x, \frac{1}{x}\right\}, \forall x \in (0, \infty)$. Here $= x^{-1}$ is the multiplicative inverse of x. One can guess and derive basic properties of the multiplicative modulus function. Some properties are the followings: $|x|_\times \geq 1, \forall x \in (0, \infty)$. $|x|_\times = 1$ if and only if $x = 1$ in $(0, \infty)$. $|x^{-1}|_\times = |x|_\times, \forall x \in (0, \infty)$. In general, $|x^\alpha|_\times = |x|_\times^{|\alpha|}, \forall x \in (0, \infty), \forall \alpha \in \mathbb{R}$. $|xy|_\times \leq |x|_\times |y|_\times, \forall x, y \in (0, \infty)$. Let us use the "log" or the "Log" notations for the natural logarithm with respect to the base e. $\log|x|_\times = \max\{\log x, -\log x\} = |\log x|, \forall x \in (0, \infty)$. $e^{|x|} = e^{\max\{x, -x\}} = \max\{e^x, e^{-x}\} = |e^x|_\times, \forall x \in \mathbb{R}$. Thus, the function $x \to e^x$ maps the additive group $\mathbb{R}$ onto the multiplicative group $(0, \infty)$, which preserves group operations and also preserves the modulus functions. The inverse-logarithm function also has the same properties. When "addition" is replaced by "multiplication", multiplicative mathematics is obtained. However, the word "multiplicative" is also used in different contexts in some articles [8, 20]. Many branches of mathematics have already been extended to multiplicative branches in the sense of replacing "addition" by "multiplication". For example, see [2, 3, 4, 15, 16, 17, 18], in connection with numerical methods, geometry, calculus, and metric spaces. There are article in connection with fractional calculus, integral transformations, and differential equations in the "multiplicative" sense. Functional analysis has been chosen in this article in terms of the "multiplicative" sense. It is known that some parts of metric space theory, topology, real analysis, complex analysis and algebra are needed for functional analysis. Let us begin with the known metric space theory in the multiplicative sense.

2. METRIC SPACE THEORY

Many generalized metrics were introduced just for fixed point theory. This class of generalized metrics also includes "multiplicative" metrics. Let us first recall the definition of a semi-metric on a set X. The terminology was used by V.K. Balachandran in his book "Topological Algebras". A general reference for this section is [17].

Let X be a non-empty set. A function $d: X \times X \to [0, \infty)$ is said to be a semi-metric, if it satisfies the following three axioms: (i) $d(x, x) = 0, \forall x \in X$. (ii) $d(x, y) = d(y, x), \forall x, y \in X$. (iii) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$. It is called a metric, if it also satisfies the condition: (iv) If $d(x, y) = 0$ in X, then $x = y$. The addition sign $+$ in (iii) brings semi-metrics under "additive" mathematics. Let d be a semi-metric on X so that (X, d) is a semi-metric space. For a given $x \in X$, and for a given $r > 0$, let $B(x, r) = \{y \in X: d(x, y) < r\}$. Consider $B(x_1, r_1), B(x_2, r_2), \ldots, B(x_n, r_n)$ in X, for some n, such that their intersection B is non-empty. It is known that for every given $z \in B$, there is a $r_z > 0$ such that $B(z, r_z) \subseteq B$. Using this fact, it follows that the collection τ of all possible unions of sets of the form $B(x, r)$ form a topology in the following sense: (a) $\emptyset \in \tau$, (b) τ is closed under finite intersections, and (c) τ is closed under arbitrary unions.

Moreover, if d is a metric, then for given $x \neq y$ in X, it is true that $B(x,r) \cap B(y,r) = \emptyset$, with $r = d(x,y)/2$, so that the topology τ is Hausdorff. The topology τ is called the topology induced by the semi-metric d on X.

Let us next extend these ideas to "multiplicative "form.

Let X be a non-empty set. A function $D: X \times X \rightarrow [1, \infty)$ is said to be a m-semi-metric, if it satisfies the following three axioms. (i) $D(x,x) = 1, \forall x \in X$. (ii) $D(x,y) = D(y,x), \forall x,y \in X$. (iii) $D(x,y) \leq D(x,z).D(z,y), \forall x,y,z \in X$. It is called a m-metric, if it also satisfies the condition: (iv) If $D(x,y) = 1$ in X, then $x = y$. The multiplication sign **.** in (iii) brings m-semi-metrics under "multiplicative" mathematics. Let D be a m-semi-metric on X so that (X, D) is a m-semi-metric space. For a given $x \in X$, and for a given $r > 1$, let $B(x,r) = \{y \in X: D(x,y) < r\}$. Consider $B(x_1, r_1), B(x_2, r_2), \dots, B(x_n, r_n)$ in X, for some n, such that their intersection B is non-empty. Let $z \in B$. Let $r_z = \min\left\{\frac{r_i}{D(x_i,z)}: i = 1, 2, \dots, n\right\} > 1$. If $x \in B(z, r_z)$, then $D(x_i, x) \leq D(x_i, z)D(z, x) < D(x_i, z)r_z \leq r_i, \forall i$. Thus, $B(z, r_z) \subseteq B$. Let τ be the collection of possible unions of sets of the form $B(x,r)$. Then τ is a topology on X. Moreover, if D is a m-metric, then for given $x \neq y$ in X, it is true that $B(x,r) \cap B(y,r) = \emptyset$, with $r = \sqrt{D(x,y)}$, so that the topology τ is Hausdorff. The topology τ is called the topology induced by the m-semi-metric D on X.

Let us now compare these two semi-metrics in terms of the exponential function and the logarithmic function.

Let X be a non-empty set. If d is a semi-metric on X, then the composition function defined by $D(x,y) = e^{d(x,y)}$ is a m-semi-metric on X, such that both topologies induced by d and D on X are equal. If D is a m-semi-metric on X, then the composition function $d = \log D$ defined by $d(x,y) = \log D(x,y)$ is a semi-metric on X, such that both topologies induced by d and D on X are equal. A sequence $(x_n)_{n=1}^{\infty}$ in a semi-metric space (X, d) is a Cauchy sequence, if for given $\varepsilon > 0$, there is an integer n_0 such that $d(x_n, x_m) \leq \varepsilon, \forall m, n \geq n_0$. A sequence $(x_n)_{n=1}^{\infty}$ in a semi-metric space (X, d) is said to converge to a point x in X, if for given $\varepsilon > 0$, there is an integer n_0 such that $d(x_n, x) \leq \varepsilon, \forall n \geq n_0$. A semi-metric space is said to be complete, if every Cauchy sequence converges to some point in that semi-metric space. A sequence $(x_n)_{n=1}^{\infty}$ in a m-semi-metric space (X, D) is a Cauchy sequence, if for given $\varepsilon > 1$, there is an integer n_0 such that $D(x_n, x_m) \leq \varepsilon, \forall m, n \geq n_0$. A sequence $(x_n)_{n=1}^{\infty}$ in a m-semi-metric space (X, D) is said to converge to a point x in X, if for given $\varepsilon > 1$, there is an integer n_0 such that $D(x_n, x) \leq \varepsilon, \forall n \geq n_0$. A m-semi-metric space is said to be complete, if every Cauchy sequence converges to some point in that m-semi-metric space. If d is a semi-metric on X such that (X, d) is complete, then for the m-semi-metric $D = e^d$, the space (X, D) is also complete, and it is conversely true. If (X, D) is a m-metric space, and if it is a topologically dense subspace of a complete m-metric space (Y, D') such that the restriction of D' to $X \times X$ is equal to D, then (Y, D') is called a (the) m-completion of (X, D). The existence and uniqueness of such completions also follow from exponential-logarithmic transformations. Unfortunately, all these exponential-logarithmic transformations have limitations in establishing some important results. Such a result is discussed in the next particular example.

Let $X = \mathbb{R}$ and $Y = (0, \infty)$. Define $d(x, y) = |x - y|$ on X. Define $D(x, y) = \left|\frac{x}{y}\right|_{\times}$ on Y. It is known that the metric space (X, d) is complete by means of the construction of all real numbers. Let us now verify that the m-metric space (Y, D) is also complete. For this purpose, let us consider a Cauchy sequence $(x_n)_{n=1}^{\infty}$ in (Y, D). Then there is an integer n_0 such that $D(x_n, x_m) \leq 2, \forall m, n \geq n_0$. Then $|x_n|_{\times} \leq \left|\frac{x_n}{x_{n_0}}\right|_{\times} |x_{n_0}|_{\times} \leq 2|x_{n_0}|_{\times}, \forall n \geq n_0$. So, there is a positive integer M such that $|x_n|_{\times} < M, \forall n$. Then $x_n > \frac{1}{M}, \forall n$. Fix $\varepsilon > 0$. Let $\varepsilon' = 1 + \frac{\varepsilon}{M}$. Then there is an integer m_0 such that $\left|\frac{x_m}{x_n}\right|_{\times} < \varepsilon' = 1 + \frac{\varepsilon}{M}, \forall m, n \geq m_0$. Then $\frac{x_m}{x_n} - 1 < \frac{\varepsilon}{M}, \forall m, n \geq m_0$ and $\frac{x_n}{x_m} - 1 < \frac{\varepsilon}{M}, \forall m, n \geq m_0$. If $n, m \geq m_0$, and if $x_n - x_m \geq 0$, then $|x_n - x_m| = x_n - x_m = \left(\frac{x_n}{x_m} - 1\right) x_m < \frac{\varepsilon}{M} M = \varepsilon$. Similarly, if $n, m \geq m_0$, and if $x_m - x_n \geq 0$, then $|x_n - x_m| = x_m - x_n = \left(\frac{x_m}{x_n} - 1\right) x_n < \frac{\varepsilon}{M} M = \varepsilon$. This means that $(x_n)_{n=1}^{\infty}$ is also a Cauchy sequence in (X, d), and let it converge to some $x \geq \frac{1}{M} > 0$ in (X, d). If the previous notations are followed, then $|x_n - x| \leq \varepsilon$ and $\left|\frac{x_n}{x}\right|_{\times} \leq \varepsilon'$, whenever $n \geq n_0$. Since there is a possibility of getting ε from given ε', it can be concluded that the sequence $(x_n)_{n=1}^{\infty}$ converges to x in (Y, D). So, (Y, D) is a complete m-metric space. Let us note that this important result follows only from the construction of the real line.

Let us observe two algebraic operations related to the open ray $(0, \infty)$. If $x, y \in (0, \infty)$, then its multiplication $x.y \in (0, \infty)$. If $x \in (0, \infty)$ and if $\alpha \in \mathbb{R}$, then the number $x^{\alpha} \in (0, \infty)$. One can check that $(0, \infty)$ is a vector space over the real field $\mathbb{R}$ according to the definition presented in the next section. Defining x^{α} in a unique way for a given complex number α is a difficult thing. This impossibility provides a "limitation" for multiplicative functional analysis.

3. VECTOR SPACES WITH OPERATION IN MULTIPLICATION FORM

The usual definition for vector spaces is to be presented here with multiplication operation for group operation. This view is essential for the development of multiplicative functional analysis. Different views are also helpful for understanding the structures. For example, there are many ways just to define a single concept of Stone Cech compactification [21]. There are many ways to define points at infinity [19]. Different views are helpful in constructing examples. There is a very simple example for a Banach space to provide an example for a fixed point-free non-expansive mappings on a weakly compact convex set [1]. There is a complicated example of a separable Banach space without a Schauder basis [9]. Both examples were famous examples because they were counter examples for two famous open problems. The vector spaces to be considered in this article are over the real field, for one reason mentioned at the end of the previous section. It is sometimes being difficult to discuss or study algebraic structures over the real field, for example, see [10]. It is sometimes being easy to discuss or study algebraic structures over the real field, for example, see [12, 23]. Since it is being difficult to define powers with complex exponent, there is a need to restrict the study of multiplicative functional analysis only over the real field. In view of the powers, it is further being difficult to define inner products [7].

Let X be a non-empty set. Let $M: X \times X \to X$ and $E: \mathbb{R} \times X \to X$ be two mappings (operations). Let us denote $M(x, y)$ by $x.y$ or simply by xy and $E(\alpha, x)$ by x^{α}, for any $x, y \in X$, and for any $\alpha \in \mathbb{R}$. The set X along with the operations M and E is denoted by (X, M, E), and it is called a vector space (over the real field), when the following properties are true.

I. $x(yz) = (xy)z, \forall x, y, z \in X$.

II. There is a unique element $1 \in X$ such that $x.1 = 1.x = x, \forall x \in X$.

III. For every $x \in X$, there is a unique element $y \in X$ such that $xy = yx = 1$.

IV. $xy = yx, \forall x, y \in X$.

V. $x^{(\alpha+\beta)} = x^{\alpha}x^{\beta}, \forall x \in X, \forall \alpha, \beta \in \mathbb{R}$.

VI. $(xy)^{\alpha} = x^{\alpha}y^{\alpha}, \forall x, y \in X, \forall \alpha \in \mathbb{R}$.

VII. $x^{(\alpha\beta)} = (x^{\alpha})^{\beta} = (x^{\beta})^{\alpha}, \forall x \in X, \forall \alpha, \beta \in \mathbb{R}$.

VIII. $x^1 = x, \forall x \in X$, for the multiplicative identity element 1 in the real field $\mathbb{R}$.

Let us use the notation 1 for both multiplicative identity elements in X and in $\mathbb{R}$. Let us use the notations x^{-1} or $\frac{1}{x}$ for the unique element y mentioned in (iii), without violating other axioms. Let us use also the notation $\frac{x}{y}$ for xy^{-1}.

Let us next observe the changes in notations of some definitions according to the notations used in the definition for vector spaces.

Let $T: X \to Y$ be a mapping from a vector space X over the real field to a vector space Y over the real field. It is said to be a linear mapping, if $T(x^{\alpha}y^{\beta}) = (T(x))^{\alpha}(T(y))^{\beta}, \forall x, y \in X, \forall \alpha, \beta \in \mathbb{R}$. Let us also use the notation Tx for $T(x)$. Moreover, T is called a linear functional, if $Y = (0, \infty)$ with the usual operations M and E.

Let X be a vector space over the real field. A subset F of X is said to be convex, if $x^{\alpha}y^{1-\alpha} \in F$, whenever $x, y \in F$, and whenever $\alpha \in \mathbb{R}$ satisfies $0 \leq \alpha \leq 1$. A subset F of X is said to be balanced, if $x^{\alpha} \in F$, whenever $x \in X$ and whenever $\alpha \in \mathbb{R}$ satisfy $|\alpha| \leq 1$. A subset Y of X is a vector subspace, if $x^{\alpha}y^{\beta} \in X$, whenever $x, y \in X$, and whenever $\alpha, \beta \in \mathbb{R}$.

4. MULTIPLICATIVE NORMS

Let us recall the definition of seminorms from the book, Functional analysis, of W. Rudin.
Let (X, M, E) be a vector space over the real field, defined in the previous section. Let $p: X \to [0, \infty)$ be a function that satisfies the conditions: (i) $p(1) = 0$; (ii) $p(x^{\alpha}) = |\alpha| p(x), \forall x \in X, \forall \alpha \in \mathbb{R}$; (iii) $p(xy) \leq p(x) + p(y), \forall x, y \in X$. Then p is called a semi-norm on X. It is called a norm on X, if it also satisfies one more condition: (iv) If $p(x) = 0$ then $x = 1$in X.

Let p be a semi-norm on a real vector space (X, M, E). Define $d(x, y) = p(xy^{-1}), \forall x, y \in X$. Then d is a semi-metric on X such that $d(xy, xz) = d(y, z), \forall x, y \in X$. Let τ_p be the topology induced by this semi-metric d on X. Then τ_p is called also as the topology induced by the semi-norm p on X. It is known that τ_p is a vector topology on X in the following sense. The operations $M: X \times X \to X$ and $E: \mathbb{R} \times X \to X$ are continuous functions, when the product topologies are considered on the domains. Here $\mathbb{R}$ is considered with the usual topology, and X is considered with the topology τ_p. Continuity of these operations can also be described in terms of sequences, in view of the semi-metrizability of the domain topologies.

Moreover, if a sequence $(x_n)_{n=1}^{\infty}$ converges to some x in (X, τ_p), then the sequence $\left(p(x_n)\right)_{n=1}^{\infty}$ converges to $p(x)$. If p is also a norm, then τ_p is Hausdorff, because d is a metric in this case.

Let us now define m-semi-norms in the multiplicative sense.

Let (X, M, E) be a vector space over the real field. Let $q: X \to [1, \infty)$ be a function that satisfies the conditions: (i) $q(1) = 1$; (ii) $q(x^{\alpha}) = (q(x))^{|\alpha|}, \forall x \in X, \forall \alpha \in \mathbb{R}$; (iii) $q(xy) \le q(x).q(y), \forall x, y \in X$. Then q is called a m-semi-norm on X. It is called a m-norm on X, if it also satisfies one more condition: (iv) If $q(x) = 1$ then $x = 1$in X.

Define $D(x, y) = q(xy^{-1}), \forall x, y \in X$. Then D is a m-semi-metric on X such that $D(xy, xz) = D(y, z), \forall x, y \in X$. Let τ_q be the topology induced by this m-semi-metric D on X. Then τ_q is called also as the topology induced by the m-semi-norm q on X. If $p(x) = \log q(x), \forall x \in X$, then p is a semi-norm on X such that $\tau_p = \tau_q$. In particular, τ_q is also a vector topology. Also, if a sequence $(x_n)_{n=1}^{\infty}$ converges to some x in (X, τ_q), then the sequence $\left(q(x_n)\right)_{n=1}^{\infty}$ converges to $q(x)$. Moreover, for a sequence $(x_n)_{n=1}^{\infty}$ in X, there is a constant $K > 1$ such that $q(x_n) \le K, \forall n$ if and only if there is a constant $L > 0$ such that $p(x_n) \le L, \forall n$. Such sequences are bounded sequences Since $\tau_p = \tau_q$, bounded sets with respect to τ_p are bounded sets with respect to τ_q. If q is also a m-norm, then τ_q is Hausdorff, because D is a m-metric in this case. Implication: If a real vector space with a m-semi-norm is considered as a topological vector space, then there is no need to develop a new theory for the multiplicative case. However, there is a need for careful changes in notations.

Let q be a m-semi-norm on a real vector space (X, M, E). Suppose $p(x) = \log q(x), \forall x \in X$. Let $(x_n)_{n=1}^{\infty}$ be a sequence which converges in $(X, \tau_q) = (X, \tau_p)$. Then this sequence should be a bounded sequence. That is, there is a constant $K > 1$ such that $q(x_n) \le K, \forall n$ and there is a constant $L > 0$ such that $p(x_n) \le L, \forall n$. This is noticed now for an aim to define an operator norm corresponding to two given m-norms. Let us use the common notation; $\|.\|$ for a m-norm in the domain and for a m-norm in the co-domain in the next proposition.

Proposition:

Let $T: X \to Y$ be a linear mapping from a m-normed space X to a m-normed space Y. Then the following are equivalent.

a) T is continuous on X.
b) T is continuous at 1.
c) T is continuous at some point of X.
d) $\|Tx\| \le K_r$ whenever $\|x\| < r$, for some $K_r \ge 1$,for every $r > 1$.
e) $\|Tx\| \le K$ whenever $\|x\| < r$, for some $K \ge 1$, for some $r > 1$.
f) If $(x_n)_{n=1}^{\infty}$ is a bounded sequence in X, then $(Tx_n)_{n=1}^{\infty}$ is a bounded sequence in Y.
g) $\|Tx\| \le \|x\|^L, \forall x \in X$, for some $L \ge 0$.

Proof: Note that $(i) \Rightarrow (ii)$, and $(i) \Rightarrow (iii)$ are true. Suppose (iii) is true at a point $x \in X$. Let us consider another point x' in X. Let $(x'_n)_{n=1}^{\infty}$ be a sequence in X such that it converges to x'. Then $x'_n x'^{-1} x \to x$ as $n \to \infty$. So, $T(x'_n x'^{-1} x) \to T(x)$ as $n \to \infty$. That is, $T(x'_n)\left(T(x')\right)^{-1} T(x) \to T(x)$, as $n \to \infty$. This proves that $T(x'_n) \to T(x')$ as $n \to \infty$. That is, T is continuous at x', for every $x' \in X$. Thus $(iii) \Rightarrow (i)$ and hence $(ii) \Rightarrow (i)$ are true.

Suppose (i) is true. To prove (vi), let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $\|x_n\| \le K, \forall n$, for some $K \ge 1$. If $(Tx_n)_{n=1}^{\infty}$ is not bounded, then by passing to a subsequence of $(x_n)_{n=1}^{\infty}$, let us assume that $\|Tx_n\| \ge n^n, \forall n$. Let $x'_n = (x_n)^{\frac{1}{n}}, \forall n$. Then $1 \le \|x'_n\| \le K^{\frac{1}{n}} \to 1$, as $n \to \infty$. But, $\|Tx'_n\| = \|Tx_n\|^{\frac{1}{n}} \ge n, \forall n$. Thus, $x'_n \to 1$ as $n \to \infty$, but $Tx'_n \nrightarrow 1 = T(1)$ as $n \to \infty$. That is, T is not continuous at 1. So, $(i) \Rightarrow (vi)$ is true.

Suppose (vi) is true. Suppose (i) is not true so that (ii) is also not true. Then there is a sequence $(x_n)_{n=1}^{\infty}$ in X such that $\|x_n\| \to 1$ as $n \to \infty$ and such that $\|Tx_n\| > K > 1, \forall n$, for some $K > 1$. By passing to a subsequence, let us assume that $\|x_n\| < 2^{\frac{1}{n}}, \forall n$. Let $x'_n = (x_n)^n, \forall n$. Then $\|x'_n\| < 2, \forall n$, and $\|Tx'_n\| = \|Tx_n\|^n > K^n \to \infty$ as $n \to \infty$. So, (vi) is not true. This proves that $(vi) \Rightarrow (i)$ is true.

Suppose (vi) is true. If (iv) is not true, then there is a sequence $(x_n)_{n=1}^{\infty}$ in X such that $\|x_n\| < r, \forall n$ for some $r > 1$ and such that $\|Tx_n\| > n, \forall n$. Thus $(vi) \Rightarrow (iv)$ and hence $(vi) \Rightarrow (v)$ are true.

Suppose (v) is true. If (vi) is not true, then there is a sequence $(x_n)_{n=1}^{\infty}$ in X such that $\|x_n\| < t, \forall n$ for some $t > 1$ and such that $\|Tx_n\| \to \infty$, as $n \to \infty$. Suppose $t^s < r$, for some $s > 0$. Let $x'_n = (x_n)^s, \forall n$, so that $\|x'_n\| < r, \forall n$ and $\|Tx'_n\| \to \infty$ as $n \to \infty$. That is, (v) is not true. Thus $(iv) \Rightarrow (v) \Rightarrow (vi)$ is true.

Suppose (i) is true. Then there is a $K > 1$ such that $\|Tx\| \le K$ whenever $\|x\| \le e$ in X. For given $x \ne 1$ in X, find a $\alpha \in \mathbb{R}$ such that $\|x^\alpha\| = e^1 = \|x\|^{|\alpha|}$. Then $\|Tx^\alpha\| \le K$ so that $\|Tx\| \le K^{\frac{1}{|\alpha|}} = e^{\frac{1}{|\alpha|}\log K} = \|x^\alpha\|^{\frac{1}{|\alpha|}\log K} = \|x\|^{\log K}$. Thus, $\|Tx\| \le \|x\|^{\log K}, \forall x \in X$. Here . This proves (vii). Thus $(i) \Rightarrow (vii)$ is true. On the other hand $(vii) \Rightarrow (ii) \Rightarrow (i)$ is true. Now, the proof of the proposition is complete.

Corollary:
Let X be a real vector space. Suppose $\|.\|_1$ and $\|.\|_2$ be two m-norms which induce a common topology τ on X. Then there are constants $K > 0$ and $L > 0$ such that $(\|x\|_1)^K \le \|x\|_2 \le (\|x\|_1)^L, \forall x \in X$.

Proof: Apply the previous proposition to the continuous identity mappings $I: (X, \|.\|_1) \to (X, \|.\|_2)$ and $I: (X, \|.\|_2) \to (X, \|.\|_1)$.

Note that the converse of the previous corollary is also true. Let us now define an operator norm for operators.

Let $T: X \to Y$ be a continuous linear transformation from a m-normed space X to a m-normed space Y. Let us call the smallest $L \ge 0$ satisfying $\|Tx\| \le \|x\|^L, \forall x \in X$, as the norm of T, and let us denote it by $\|T\|$. Note that $\|Tx\| \le \|x\|^{\|T\|}, \forall x \in X$ is also true.

If $(X, \|.\|)$ is a m-normed space, then it is a topologically dense vector subspace of a complete m-normed space $(X', \|.\|')$ such that the restriction of $\|.\|'$ to X is equal to $\|.\|$, and then $(X', \|.\|')$ is called a (the) m-completion of $(X, \|.\|)$. The existence and uniqueness of such completions also follow from exponential-logarithmic transformations. If X' and Y' are the completions of two m-normed spaces X and Y, respectively, and if $T: X \rightarrow Y$ is a continuous linear transformation from the m-normed space X to the m-normed space Y, then there is a unique continuous linear transformation $T': X' \rightarrow Y'$ such that its restriction to X is T. Moreover, $\|T'\| = \|T\|$. These facts can be verified directly without using exponential-logarithmic transformation.

Let $T: X \rightarrow Y, T_1: X \rightarrow Y$ and $T_2: X \rightarrow Y$ be continuous linear transformations from a m-normed space X to a m-normed space Y. Let α be a real number. Define $T^\alpha: X \rightarrow Y$ and $T_1T_2: X \rightarrow Y$ by $T^\alpha(x) = (Tx)^\alpha, (T_1T_2)(x) = (T_1(x))(T_2(x)), \forall x \in X$. Then these two linear transformations are continuous linear transformations and then $\|T_1T_2\| \leq \|T_1\| + \|T_2\|$ and $\|T^\alpha\| = \|T\|^{|\alpha|}$. Moreover, if $\|T\| = 0$, then $Tx = 1, \forall x \in X$. Thus the operator norm is a norm, but not a m-norm. If $S: Y \rightarrow Z$ is another continuous linear transformation into a m-normed space Z, then $S \circ T: X \rightarrow Z$ is also a continuous linear transformation and then $\|S \circ T\| \leq \|S\|.\|T\|$. Let us note that T is a linear functional when $Y = (0, \infty)$.

It is possible to define compact linear transformations and derive results by using bounded sequences and exponential-logarithmic transformations. Some Hahn-Banach theorems are based on semi-norms as they can be seen from the third chapter of the book, Functional Analysis, of W. Rudin. Let us discuss them in the next section, because m-normed spaces are topological vector spaces.

5. TOPOLOGICAL VECTOR SPACES

There is no need to have a separate study for topological vector spaces, in "multiplicative" sense. Since there are metrics, conversions of results to m-metrics should be done with a due respect to notations. All the results given in the second chapter of the book, Functional Analysis, of W. Rudin can be converted to m-metrics through exponential-logarithmic transformations. Let us first concentrate on the metrization theorem provided in the first chapter of that book.

The metrization theorem:
Let X be a real Hausdorff topological vector space with a countable local base at 1. Then there is a m-metric D on X with the following properties:

(a) The topology of X is induced by D.
(b) The open balls of the form $\{x \in X: D(1, x) < r\}$ are balanced.
(c) $D(xz, yz) = D(x, y), \forall x, y, z \in X$.

If, in addition, X is locally convex, then D can be chosen so that all balls given in (b) are convex, and D sastisfies (a), (b), and (c).

Proof: By the metrization theorem given in the first chapter of the book, Functional Analysis, of W. Rudin, there is a metric d on X with the following properties: (a) The topology of X is induced by d; (b) The open balls of the form $\{x \in X: d(1, x) < r\}$ are balanced; and (c) $d(xz, yz) = d(x, y), \forall x, y, z \in X$. If X is also locally convex, then d can be chosen so that all balls given in (b) are convex, and d sastisfies (a), (b), and (c). Define $D(x, y) = e^{d(x,y)}, \forall x, y \in X$. Then D is a required m-metric on X, and the proof is complete.

Note that if D is a m-metric which induces the topology in a real Hausdorff topological vector space, then there a countable local base at 1. Let us next illustrate a conversion of a Hahn-Banach theorem given in the third chapter of the book, Functional Analysis, of W. Rudin.

A Hahn-Banach Theorem:
Suppose Y is a vector subspace of a real vector space X, q is a m-seminorm on X, and $g: Y \to (0, \infty)$ is a linear functional such that $|g(x)|_\times \leq q(x), \forall x \in Y$. Then there is a linear functional $G: X \to (0, \infty)$ such that the restriction G to Y is g and such that $|G(x)|_\times \leq q(x), \forall x \in X$.

Proof: Let us define $p(x) = \log q(x), \forall x \in X$ and let us define $f(x) = \log g(x), \forall x \in Y$. Then $p: X \to [0, \infty)$ is a seminorm on X and $f: Y \to \mathbb{R}$ is a linear functional in the sense that $p(xy) \leq p(x) + p(y), \forall x, y \in X$ and $f(x^\alpha y^\beta) = \alpha f(x) + \beta f(y), \forall x, y \in Y, \forall \alpha, \beta \in \mathbb{R}$. Moreover, $f(x) \leq p(x), \forall x \in Y$. Then by a known Hahn-Banach theorem, there is a linear functional $F: X \to \mathbb{R}$ such that the restriction of F to Y is f and such that $F(x) \leq p(x), \forall x \in X$. Let us define $G(x) = e^{\log F(x)}, \forall x \in X$. Then $G: X \to (0, \infty)$ is a linear functional such that the restriction of G to Y is g, and such that $G(x) \leq q(x), \forall x \in X$. Since $(G(x))^{-1} = G(x^{-1}) \leq q(x^{-1}) = q(x), \forall x \in X$, then $|G(x)|_\times \leq q(x), \forall x \in X$. This proves the theorem.

Corollary:
Suppose Y is a vector subspace of a real m-normed space X and $g: Y \to (0, \infty)$ is a continuous linear functional with norm $\|g\|$. Then there is a linear functional $G: X \to (0, \infty)$ such that the restriction G to Y is g and such that $\|G\| = \|g\|$.

Proof: Without loss of generality, let us assume that $\|g\| > 0$. Let $L = \|g\|$. Define $f: Y \to (0, \infty)$ by $f(x) = (g(x))^{\frac{1}{L}}, \forall x \in Y$. Then $|f(x)|_\times \leq \|x\|, \forall x \in Y$. By the previous theorem, there is a linear functional $F: X \to (0, \infty)$ such that the restriction of F to Y is f and such that $|F(x)|_\times \leq \|x\|, \forall x \in X$. Define $G: X \to (0, \infty)$ by $G(x) = (F(x))^L, \forall x \in X$. This G is a required liner functional.

6. EXAMPLES

Only one example about $(0, \infty)$ has been discussed in the first section. Before discussing a few more examples, let us note that the corollary given in the fourth section states that all m-norms in any finite dimensional real vector space induce a common vector topology, because any finite dimensional real vector space has a unique Hausdorff vector topology. The completeness of $(0, \infty)$ with respect to $|.|_\times$ was established in the second section. This is also needed in this section.

Example 1:
Let us fix a natural number n. Let $X = X_1 \times X_2 \times \ldots \times X_n$, where each $X_i = (0, \infty)$. Define $M: X \times X \to X$ by $M((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$, whenever $x_i, y_i \in (0, \infty), \forall i$. Define $E: \mathbb{R} \times X \to X$ by $E(\alpha, (x_1, x_2, \ldots, x_n)) = ((x_1)^\alpha, (x_2)^\alpha, \ldots, (x_n)^\alpha)$, whenever $\alpha \in \mathbb{R}$ and $(x_1, x_2, \ldots, x_n) \in X$. Define $\|(x_1, x_2, \ldots, x_n)\|_1 = \prod_{i=1}^n |x_i|_\times, \forall (x_1, x_2, \ldots, x_n) \in X$. Define $\|(x_1, x_2, \ldots, x_n)\|_\infty = \max\{|x_i|_\times : i = 1, 2, \ldots, n\}$. Define $D_1((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \prod_{i=1}^n \left|\frac{x_i}{y_i}\right|_\times, \forall (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X$.

Define $D_\infty\big((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)\big) = \max\left\{\left|\frac{x_i}{y_i}\right|_\times : i = 1,2,\dots,n\right\}, \forall (x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n) \in X$. Then (X,M,E) is a vector space over the real field. $\|.\|_1$ and $\|.\|_\infty$ are two m-norms on X inducing the m-metrics D_1 and D_∞, respectively. Both of them induce a common vector topology which coincides with the subspace topology induced by the usual topology on $\mathbb{R}^n$. Fix a real number α. Define $T: X \to X$ by $T\big((x_1,x_2,\dots,x_n)\big) = ((x_1)^\alpha,(x_2)^\alpha,\dots,(x_n)^\alpha), \forall (x_1,x_2,\dots,x_n) \in X$. Then, T is a linear transformation such that $\|T\| = |\alpha|$, when both domain and codomain are endowed with $\|.\|_1$ or both with $\|.\|_\infty$

Example 2:
Let $X = \left\{(a_n)_{n=1}^\infty : a_n \in (0,\infty), \forall n, \prod_{i=1}^\infty |a_n|_\times \; converges \; in \; \big((0,\infty),|.|_\times\big)\right\}$. Define $M((a_n)_{n=1}^\infty,(b_n)_{n=1}^\infty) = (a_n b_n)_{n=1}^\infty$ and $E(\alpha,(a_n)_{n=1}^\infty) = ((a_n)^\alpha)_{n=1}^\infty$, for $(a_n)_{n=1}^\infty,(b_n)_{n=1}^\infty \in X$, and for $\alpha \in \mathbb{R}$. Define $\|(a_n)_{n=1}^\infty\|_1 = \prod_{n=1}^\infty |a_n|_\times, \forall (a_n)_{n=1}^\infty \in X$. Then (X,M,E) is a complete m-normed space with the m-norm $\|.\|_1$. Define a linear functional $f: X \to (0,\infty)$ by $f((a_n)_{n=1}^\infty) = \prod_{n=1}^\infty a_n, \forall (a_n)_{n=1}^\infty \in X$. Then f is continuous and the operator norm $\|f\| = 1$. See [15] for results in connection with infinite products.

Example 3:
Let $X = \{(a_n)_{n=1}^\infty : a_n \in (0,\infty), \forall n, |a_i|_\times \to 1 \; as \; i \to \infty\}$. Define M and E as in the previous Example 2. Define $\|(a_n)_{n=1}^\infty\|_\infty = \sup\{|a_n|_\times : n = 1,2,3,\dots.\}$. Then (X,M,E) is a complete m-normed space with the m-norm $\|.\|_\infty$. Define a linear functional $f: X \to (0,\infty)$ by $f((a_n)_{n=1}^\infty) = a_7, \forall (a_n)_{n=1}^\infty \in X$. Then f is continuous and the operator norm $\|f\| = 1$.

These two examples provide "sequence spaces" [11] and then "Schauder bases", and then a possibility to generalize these bases [13].

7. LEBESGUE MEASURABLE SETS

One may find multiplicative measures and corresponding integrations in the articles [16, 18]. Theory discussed so far and Example 1 given in the previous section give a possibility to define Lebesgue measurable sets on the space X given in Example 1. It is not tried to define Lebesgue measure on this space, because it is being complicated to define a multiplicative Euclidean norm on that space. But, Lebesbue measurable sets in that space form a unique collection that is independent of the m-norms on that space in view of the Corollary given in the fourth section, and in view of construction of Hausdorff measures [5, 6, 22]. However, a standard reference for this section is the book, Hausdorff Measures, of C.A. Rogers. The arguments provided in that book should be modified to get the following Lebesgue measurable sets. One may find no difficulty in extending concepts and results up to Section 3 of Chapter 2 of that book to the multiplicative form explained in the next paragraph. It seems that there is no simple extension for Theorem 40 (Section 4 of Chapter 2) of that book to the multiplicative form. Theorem 40 of that book is the required result to define Hausdorff dimension [5, 14] . So, it is being difficult to define Hausdorff dimension in a simple multiplicative form. Now, let us define Lebesgue measurable sets in the multiplicative form by following the arguments of C.A. Rogers given in his book.
Let X be a non-empty set. Let $\mathcal{P}(X)$ denote the collection of all subsets of X. A collection $\mathfrak{M}$ of subsets of X is called a σ-algebra if $\mathfrak{M}$ contains the empty set, $\mathfrak{M}$ is closed under set complements, and $\mathfrak{M}$ is closed under countable unions. $\mathcal{P}(X)$ is an example for a σ-algebra. Let D be a m-metric on X. For each subset F of X, let $Diam(F) = \sup\{D(x,y): x,y \in F\}$ with a convention that $Diam(\emptyset) = 1$. For each $\delta > 1$, and for each $F \in \mathcal{P}(X)$, let $\mu_\delta(F) = \inf\{\prod_{n=1}^\infty Diam(F_n) : F_i \subseteq X, \forall i, Diam(F_i) \le \delta, \forall i, F \subseteq \bigcup_{n=1}^\infty F_n\}$. For each $F \in \mathcal{P}(X)$, let $\mu(F) = \sup\{\mu_\delta(F): \delta > 1\}$.

Then $\mu: \mathcal{P}(X) \to [1, \infty]$ has the following properties: (i) $\mu(\emptyset) = 1$; (ii) If $F_1 \subseteq F_2 \subseteq X$, then $\mu(F_1) \leq \mu(F_2)$; and (iii) If $F_1, F_2, F_3, \ldots \in \mathcal{P}(X)$, then $\mu(\bigcup_{n=1}^{\infty} F_n) \leq \prod_{n=1}^{\infty} \mu(F_n)$. Let $\mathfrak{M}_L = \{F \subseteq X: \mu(A \cup B) = \mu(A)\mu(B), whenever\ A \subseteq F,\ B \subseteq X \backslash F\}$. Then $\mathfrak{M}_L$ is a σ-algebra that contains all closed subsets of the m-metric space (X, D). Moreover, if $F_1, F_2, F_3, \ldots$ are mutually disjoint sets in $\mathfrak{M}_L$, then $\mu(\bigcup_{n=1}^{\infty} F_n) = \prod_{n=1}^{\infty} \mu(F_n)$. If $X = (0, \infty)^n$ for some natural number n, and if D is taken as D_1 or D_∞, given in Example 1 of the previous section, then the sets in $\mathfrak{M}_L$ are called m-Lebesgue measurable sets in $(0, \infty)^n$. Both D_1 and D_∞ give the same collection of m-Lebesgue measurable sets.

8. CONCLUSIONS

Some of the limitations in developing multiplicative functional analysis are the followings. It is being difficult to extend the theory to vector spaces over complex field. It is being difficult to introduce additional structures like inner products, and to introduce algebras instead of vector spaces. It is being difficult in finding an operator m-norm for continuous linear operators from a m-normed space to a m-normed space. It is difficult to introduce m-Lebesgue measure to use it for functional analysis. It is being difficult to use exponential-logarithmic transformations in deriving some results. Apart from such difficulties, it is being easy to derive results in multiplicative functional analysis, and this provides training for logical thinking.

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