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Strong Independent Edge Saturation Number of Graphs

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ABSTRACT

Let G = (V, E) be a simple graph. Let e ∈ E(G). The strong independent edge saturation number of an edge e is defined as I^{se}(e) = max { $|S|$:S is a maximal strong independent edge dominating set of G containing e}. The Strong independent edge saturation number of the graph G is defined as I^{se}(G) = min $\{I^{se}(e): e \in E(G)\}$. In this paper the strong independent edge saturation number of some standard graphs and some corona related graphs are determined.

***Keywords*:** Saturation number, Strong domination, Strong Independent domination and edge saturation number

AMS Subject Classification (2010): 05C69

1. INDRODUCTION

Let G = (V, E) be a simple graph. The degree of any vertex u in G is the number of edges incident with u and it is denoted by deg u. The degree of the edge uv = deg u + deg v − 1 [7]. A subset S of V(G) is called the dominating set if every vertex in V – S is adjacent to at least one vertex in S. The domination number γ(G) is the minimum cardinality of a dominating set of G [8,9,11]. A subset D of E(G) of a graph G is called an edge dominating set if every edge in E − D is adjacent to atleast one edge in D. The edge domination number γ' (G) is the

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minimum cardinality of an edge dominating set of G [5,13]. An edge dominating set which is independent is called an independent edge dominating set [2]. An edge e strongly dominates another edge f if e and f are adjacent and deg e ≥ deg f. An edge dominating set D of G is called a strong edge dominating set if for every edge e_i in E – D, there exists an edge e_j in D such that $e_j$strongly dominates e_i [1,12]. A strong edge dominating set of G which is also independent is called an strong independent edge dominating set of G [1,10]. The minimum cardinality of the independent strong dominating set of G is called an independent strong domination number of G. Domsaturation number of a graph was introduced by Archarya [3,6]. The domsaturation number of a graph G is defined to be the least positive integer k such that every vertex of G lies in a dominating set of cardinality k. The concept of independence saturation number of graph was introduced by Arumugam and Subramanian [4]. Let IS (v, G) denote the maximum cardinality of an independent set in G which contains v. Then IS (G) = $\min\{\mathrm{IS}\ (\mathrm{v}, \mathrm{G}): \mathrm{v} \in \mathrm{V}\}$ is called the independence saturation number of G [4,5].

DEFINITION 1.1: The wheel graph W_n is defined to be the graph $K_1 + C_{n-1}$.

DEFINITION 1.2: The bistar graph is the graph obtained by joining the centre vertices of two copies of $K_{1,n}$ by an edge and is denoted by $B_{n,n}$.

DEFINITION 1.3: The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G by taking one copy of G_1(which has p points) and p_1 copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2.

2. MAIN RESULT

DEFINITION 2.1: Let $G = (V, E)$ be a simple graph. Let $e \in E(G)$. Let I^{se} (e) = max $\{|S: S\ is\ a\ minimal\ strong\ independent\ edge\ dominating\ set\ of\ G\ containing\ e|\}$. The strong independent edge saturation number of graph G, denoted by I^{se} (G) is defined as I^{se} (G) = min $\{I^{se}(e): e \in E(G)\}$.

REMARK 2.2: Suppose $e \in E(G)$ is not contained in any strong independent edge dominating set of G.Then $I^{se}(e) = 0$. Hence $I^{se}\ (G) = 0$.

EXAMPLE 2.3: Consider the following graph G

E(G) = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

deg e_1 = 2, deg e_2 = 3, deg e_3= 3, deg e_4 = 3, deg e_5 = 3, deg e_6 = 2, deg e_7 =2.

$S_1 = \{e_2, e_5\}$, $S_2 = \{e_3, e_5\}$, $S_3 = \{e_4, e_1, e_6\}$ and $S_4 = \{e_4, e_1, e_7\}$ are minimal strong independent edge dominating sets of G. $|S_1| = |S_2| = 2$ and $|S_3| = |S_4| = 3$.

S_3 and S_4 contain e_1. Therefore $I^{se}\ (e_1)$ = 3. S_1 contains e_2 and hence $I^{se}\ (e_2)$ = 2.

S_2 contains e_3 .Therefore I^{se} (e_3) = 2. S_3 and S_4 contains $e_{4.}$ Therefore $I^{se}(e_4)$ = 3.

S_1 and S_2 contain e_5. Therefore I^{se} (e_5) = 2. S_3 contains e_6 . Therefore I^{se} (e_6) = 3.

S_4 contains e_7 . Therefore I^{se} (e_7) = 3. Hence I^{se}(G) = min {2 , 3} = 2.

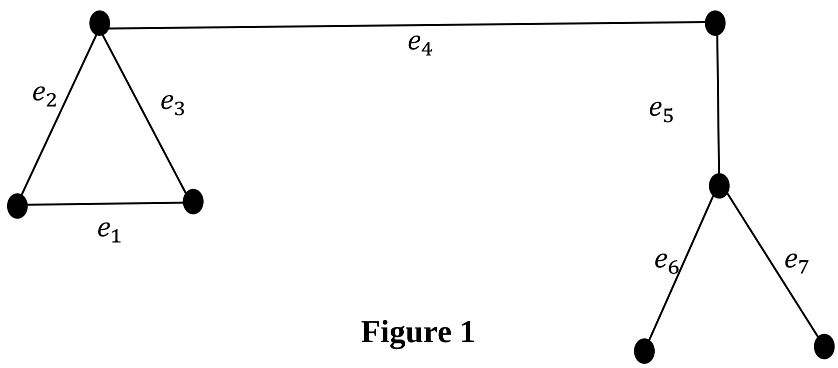


Figure 1

EXAMPLE 2.4: Consider the following graph G

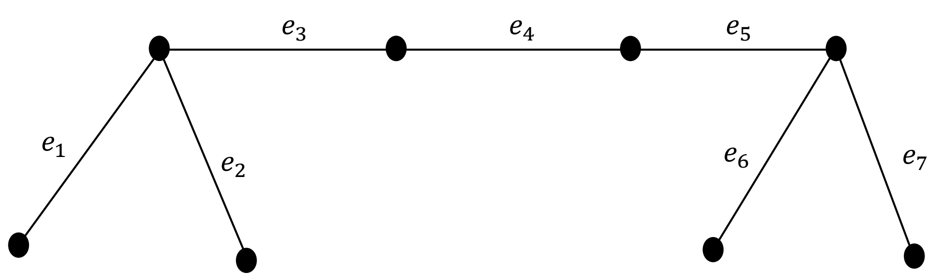


Figure 2

In the above graph G, E(G) = $\{e_{1,}\ e_{2,}\ e_{3,}\ e_{4,}\ e_{5,}\ e_{6,}\ e_7\}$

deg e_1 = deg e_2 = deg e_6 = deg e_7 = 2 , deg e_3 = deg e_5 = 3 and dege_4 = 2.

$\{e_{3\,,}\ e_5\}$ is the unique strong independent edge dominating set of G.

I^{se} (e_3) = I^{se} (e_5) = 2. Since no other strong independent edge dominating set of G exists, I^{se} (e_i) = 0, i = 1, 2, 4, 6, 7. Hence I^{se} (G) = min {0, 2} = 0.

REMARK 2.5:

Let G be a graph with $|V(G)| \geq 4$ and $|E(G)| \geq 3$.

If there is a unique strong independent edge dominating set, then I^{se} (G) = 0.

THEOREM 2.6:

Let P_n be the path with n vertices.

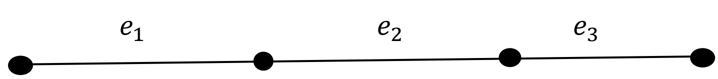
$$I^{se}(P_n) = \begin{cases} 1 & \text{if } n = 2, 3 \text{ and } 0 \text{ if } n = 4 \\ \frac{n-1}{2} & \text{if } n \geq 5 \text{ and n is odd} \\ \frac{n-2}{2} & \text{if } n \geq 6 \text{ and n is even} \end{cases}$$

Proof: Let V(G) = $\{v_1, v_2, \ldots v_n\}$ and $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$, E(G) = $\{e_i \,/\, 1 \leq i \leq n\}$

Case i: Let n = 2, $I^{se}(P_2) = 1$

Case ii: Let n = 3. $\{e_1\}$ and $\{e_2\}$ are strong independent edge dominating sets of P_3. $I^{se}(e_1) = I^{se}(e_2) = 1$. Hence $I^{se}(P_3) = 1$

Case iii: Let n = 4



$\{e_2\}$ is the unique strong independent edge dominating set of P_4. By the remark 2.2, $I^{se}(P_4) = 0$.

Case iv: When n is odd and n ≥ 5

Let $T_1 = \{e_1, e_3, e_5, \cdots, e_{n-2}\}$ and $T_2 = \{e_2, e_4, e_6, \cdots, e_{n-1}\}$. T_1 and T_2 and the minimal strong independent edge dominating sets of maximum cardinality containing e_1, e_3 e_5, $\cdots$ e_{n-2} and e_2, e_4, e_6, $\cdots$ e_{n-1} respectively. $|T_1| = \frac{n-1}{2}$ and $|T_2| = \frac{n-1}{2}$. $I^{se}(e_i) = \frac{n-1}{2}$ for all $e_i \in E(P_n)$.

Therefore $I^{se}(P_n) = \frac{n-1}{2}$, when n is odd and n ≥ 5

Case v: When n is even, n ≥ 6

$T_1 = \{e_2, e_4, e_6, \cdots, e_{n-2}\}$ and $T_2 = \{e_1, e_3, e_5, \cdots, e_{n-1}\}$ are the minimal strong independent edge dominating sets of maximum cardinality containing $e_2, e_4, e_6, \cdots, e_{n-2}$ and $e_1, e_3, e_5, \cdots, e_{n-1}$ respectively. $|T_1| = \frac{n-2}{2}$ and $|T_2| = \frac{n}{2}$. $I^{se}(e_i) = \frac{n-2}{2}$, i = 2,4,⋯ n-2 and $I^{se}(e_i) = \frac{n}{2}$, i = 1,3,5, ⋯ n – 1. Therefore $I^{se}(P_n) = \min\left\{\frac{n-2}{2}, \frac{n}{2}\right\} = \frac{n-2}{2}$.

Hence $I^{se}(P_n) = \frac{n-2}{2}$, when n is even and n ≥ 6

THEOREM 2.7:

Let G be a cycle C_n, n ≥ 3. Then $I^{se}(G) = \left\lfloor \frac{n-1}{2} \right\rfloor$

Proof:

Let V(G) = $\{v_1, v_2, \cdots v_n\}$. Let $e_i = v_i v_{n-1}$ for all 1 ≤ i ≤ n-1, $e_n = v_i v_n$ 1 ≤ i ≤ n

Let E(G) = $\{e_1, e_2, e_3, \cdots, e_n\}$

Case i: When n is odd.

Let $T_1 = \{e_1, e_3, \cdots, e_{n-2}\}$ and $T_2 = \{e_2, e_4, \cdots, e_{n-1}\}$ are minimal strong independent edge dominating sets of maximum cardinality containing $e_1, e_3, e_5, \cdots, e_{n-2}$ and $e_2, e_4, \cdots, e_{n-1}$respectively.

$|T_1| = \frac{n-1}{2} = |T_2|$. $T_3 = \{e_3, e_5, e_7, \cdots, e_n\}$ is the minimal strong independent edge dominating set containing e_n. $|T_3| = \frac{n-1}{2}$. $I^{se}(e_i) = \frac{n-1}{2}$ for all $e_i \in E(C_n)$. Therefore $I^{se}(C_n) = \min\left\{\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}\right\} = \frac{n-1}{2}$.

Case ii: When n is even, n ≥ 4

Let $T_1 = \{e_1, e_2, \cdots, e_{n-1}\}$ and $T_2 = \{e_2, e_4, \cdots, e_n\}$ are minimal strong independent edge dominating sets of maximum cardinality $e_1, e_3, \cdots, e_{n-1}$ and $e_2, e_4, \cdots, e_n$ respectively. $|T_1| = \frac{n}{2} = |T_2|$. $I^{se}(e_i) = \frac{n}{2}$ for all $e_i \in E(C_n)$.

Therefore $I^{se}(C_n) = \min\left\{\frac{n}{2}, \frac{n}{2}\right\} = \frac{n}{2}$

From case i and case ii, $I^{se}(G) = \left\lfloor \frac{n}{2} \right\rfloor$, n ≥ 3.

THEOREM 2.8:

Let G be a star $K_{1,n}$, n ≥ 1. Then $I^{se}(G) = 1$.

Proof:

Let $V(G) = \{v\, v_i / 1 \le i \le n\}$ and $E(G) = \{e_i / e_i = vv_{i,}\ 1 \le i \le n\}$

$T_i = \{e_i\}$, $1 \le i \le n$ are minimal strong independent edge dominating sets of G containing e_i respectively. $I^{se}(e_i) = 1,\ \ 1 \le i \le n$. Hence $I^{se}(G) = 1$

THEOREM 2.9:

Let G be a bistar $D_{r, s}$, r, s ≥ 1. Then I^{se}(G) = 0

Proof: Let G = $D_{r, s}$, r, s ≥ 1, r ≥ s and let u and v be the central vertices of G.

Let $u_1, u_2, \cdots, u_r$ be the vertices adjacent with u and let $v_1, v_2, \cdots, v_s$ be the vertices adjacent with v respectively. Let e = uv, $e_i = uu_i$, 1 ≤ i ≤ r and $f_j = vv_j$, 1 ≤ j ≤ s and

E(G) = $\{e,\ e_i, f_j / 1\ \leq i\ \leq r \text{ and } 1\ \leq j \leq s\}$.

deg e = r+ s, deg e_i = r and deg f_j = s ,1 ≤ i ≤ r and 1 ≤ j ≤ s. e is the unique maximum degree edge. $\{e\}$ is the unique strong independent edge dominating set of G.

Therefore $I^{se}(e)$ = 1 and $I^{se}(e_i) = I^{se}(f_j)$ = 0, 1 ≤ i ≤ r and 1 ≤ j ≤ s.

Hence I^{se}(G) = min {0, 1} = 0.

THEOREM 2.10:

Let G be a complete graph K_n, n ≥ 2. Then $I^{se}(G) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even} \\ \frac{n-1}{2} \text{ if } n \text{ is odd} \end{cases}$

Proof:

Let V(G) = $\{v_i, 1 \leq i\ \leq n\}$. There are nC_2 edges in K_n

Case i:

Let n be even. Each $\frac{n}{2}$ independent edges form a maximum strong independent edge dominating set of G. Each edge belongs to at least one of the strong independent edge dominating set. Therefore I^{se}(e) = $\frac{n}{2}$ for all e ∈ E(G) where E(G) is the edge set of G. Hence I^{se} (G) = $\frac{n}{2}$.

Case ii:

If n is odd, each $\frac{n-1}{2}$ independent edges form a strong independent edge dominating set of G. As in case (i) I^{se}(e) = $\frac{n-1}{2}$ for all e ∈ E(G). Hence I^{se} (G) = $\frac{n-1}{2}$

THEOREM 2.11:

Let G be a wheel graph W_n, n ≥ 3. Then $I^{se}(G) = \begin{cases} \frac{n-1}{2} & if\ n\ is\ odd, n \geq 5 \\ \frac{n}{2} & if\ n\ is\ even, n \geq 4 \end{cases}$

Proof:

Let V(G) = {v, v_i / 1 ≤ i ≤ n}. Let $e_i = v_i v_{i+1}$, $e_n = v_n v_i$ and $f_i = vv_i$.

Let E(G) = {e_i, f_i / 1 ≤ i ≤ n}. deg $f_i = n + 1$ and deg e_i = 4.

Let T be any strong independent edge dominating set of G. Since deg f_i ≥ deg e_i, 1 ≤ i ≤ n and the edges $f_1, f_2, \cdots f_n$ are mutually adjacent, any one f_i must belong to T. The edges which are adjacent to f_i do not belong to T. The subgraph induced by the remaining edges is P_{n-2}.

Case i: When n is odd

There are n-3 edges in P_{n-2} where n-3 is even. There are $\frac{n-3}{2}$ independent edges, since the degree of the edges e_i are same. These $\frac{n-3}{2}$ edges together with f_i form T. Hence $|T| = \frac{n-1}{2}$. No strong independent edge dominating set with cardinality greater than $\frac{n-1}{2}$ exists. I^{se}(e) = $\frac{n-1}{2}$ for all e ∈ E (W_n). Hence I^{se}(G) = $\frac{n-1}{2}$.

Case ii: When n is even

There are n-3 edges in P_{n-2} where n-3 is even n-3 is odd. There are $\frac{n-2}{2}$ independent edges, since the degree of the edges e_i are same. These $\frac{n-2}{2}$ edges together with f_i form T. Hence $|T| = \frac{n}{2}$. No strong independent edge dominating set with cardinality greater than $\frac{n}{2}$ exists. I^{se}(e) = $\frac{n}{2}$ for all e ∈ E(W_n). Hence I^{se}(G) = $\frac{n}{2}$.

THEOREM 2.12:

Let P_n be the path with n vertices. Let G = $P_n \odot K_1$ where n ≥ 1, then $I^{se}(P_n \odot K_1) = 0$.

Proof:

Let V (G) = { v_i, u_i / 1 ≤ i ≤ n} and E(G) = {$vv_{i+1}, v_i u_j$ / 1 ≤ i ≤ n-1, 1 ≤ j ≤ n}.

Let e_i= $v_i v_{i+1}$, 1 ≤i ≤ n-1. Let f_j= $v_i u_j$, 1 ≤ j ≤ n. deg e_1= deg e_{n-1}= 2, deg e_i= 3, 2 ≤ i ≤ n-2. deg f_1= deg f_n= n. deg f_j= 2, 2 ≤ j ≤ n. Let S be any strong independent edge dominating set of G.

Either e_1 ∈ S or e_2 ∈ S. Since S is independent. f_2 does not belong to S.

Hence f_2 does not belong to any strong independent edge dominating set of G.

Therefore $I^{se}(f_2)$ = 0. Hence $I^{se}(P_n \odot K_1)$ = 0.

THEOREM 2.13:

Let G = $K_{1,n} \odot K_1$, n ≥ 1, then I^{se}(G) = 0.

Proof:

Let V (G) = {v, u, v_i, u_i / 1 ≤ i ≤ n} and E(G) = {vu, $vv_i, v_i u_i$ / 1 ≤ i ≤ n}. deg uv = n,

deg vv_i = n+1 and deg $v_i u_i$ = 1, 1 ≤ i ≤ n. Let S = {$vv_i, v_j u_j$ / 1 ≤ i ≤ n, 1 ≤ j ≤ n, j ≠ i }. The above S_i, 1 ≤ i ≤ n are strong independent edge dominating sets of G. $|S_i|$ = n. No other strong independent edge dominating set exist without any vv_i. Hence I^{se} (e) = n for all e ∈ E(G). Since the edge uv does not belong to any strong independent edge dominating set of G, $I^{se}(uv)$ = 0. Therefore I^{se}(G) = 0.

OBSERVATION 2.14:

Let G = (V,E) be a simple graph. Let $|E(G)|$ = n

I^{se}(G) = n iff G ≅ nK_2, $\overline{K_n}$

I^{se}(G) = n-1 iff G ≅ P_3 ∪ (n-2) K_2

I^{se}(G) =1 iff G ≅ $K_{1,n}$.

3. CONCLUSION

In this paper Strong Independent Edge Saturation Number of some standard graph are determined. Further Strong Independent Edge Saturation Number of path related graphs, cycle related graphs and corona related graphs can be found.

References

[1] Acharya B. D., The strong domination number of a graph and related concepts, *J. Math. Phys. Sci* 14 (5) (1980) 471-475

[2] Allan R. B. and Laskar R., On domination and independent domination numbers of a graph, *Discrete Math*, 23 (1978, 73-76

[3] Arumugam S. and Kala R., Domsaturation number of a graph, *Indian J. Pure Apple. Math.* 33 (11) (2002) 1671-1676

[4] Arumugam S. & Subramanian M., Independence saturation and extended domination chain in graphs, *AKCE J. Graphs. Combin.* 4 (2) (2007) 59-69

[5] Arumugam S. and Velammal S., Edge domination in graphs, *Taiwaness Journal of Mathematics,* vol. 6, No. 2, (1998) 173-179

[6] Devadhas Nidha, Murugan Kala , A Note on edge - domsaturation number of a graph, *Open Journal of Discrete Mathematics,* 2 (2012) 109-113

[7] Harary F., (1969). Graph Theory, Adison Wesley, Reading Mass.

[8] Haynes T.W., Hedetniemi, Peter S.T., Slater J., (Eds), (1998). Domination in Graphs: Advanced Topics, Marcel Decker, Inc., New York.

[9] Haynes T.W., Hedetniemi, Peter S.T., Slater J., (Eds), (1998). Fundamental of Domination in Graphs, Marcel Decker Inc., New York.

[10] Kamath S.S., Bhat R.S., Strong (weak) independent sets and vertex coverings of a graph, *Discrete Mathematics*, 307 (2007) 1136-1145

[11] Kulli V.R., (2010). Theory of Domination in Graphs (Vishwa International Publications, Gulbarga, India.

[12] Sampathkumar E., and Pushpalatha, Strong (weak) Domination and Domination Balance in a graph, *Discrete Mathematics*, 161 (1996) 235-242

[13] Yannakakis M., and Gavril F., Edge Dominating sets in graphs, *SIAM J. APPL. Math.* 38 (1980) 364-372