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α-Partitions, α-Covers and their Vector Representations and Monoids

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ABSTRACT

This paper, besides the basics of fuzzy equivalence and fuzzy tolerance relations, primarily aims at describing methods for vector representation of equivalence classes of an α-partition and tolerance classes of an α-cover. In addition, a method for the derivation of fuzzy partitions and fuzzy covers from a given vector structure is outlined. Finally, certain monoids of α-partitions and α-covers are introduced.

Keywords**:** fuzzy equivalence relation, fuzzy tolerance relation, α-partition, α-cover, monoid

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1. INTRODUCTION

The theory of relations, specifically that of *equivalence* and *compatibility* (aka tolerance) relations, is one of the most widely studied areas of mathematics. It plays an essential role in modelling various concepts in *hard* as well as *soft* sciences. In particular, a great variety of

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binary relations have been developed for applications in decision analysis, information theory, measurement theory, social sciences, mathematical psychology, linguistics, etc.

A relation on a set which is reflexive and symmetric is called a *compatibility relation* (aka *tolerance relation*) and its transitive closure is an equivalence relation. To our knowledge, the mathematics of compatibility relations was not as widely propagated as that of equivalence relations until Kurepa's work [19] appeared. Kurepa [19] seems to be the earliest full-blown mathematical exposition on the study of reflexive symmetric relations and graphs. Since then, a large number of works dealing with the fundamentals of tolerance relations as well as their applications have appeared. It should be noted that the term *tolerance* in this context was first introduced by E. Zeeman (cf. [34]) and, in order to avoid conflicts [1], we will use tolerance in place of compatibility throughout this paper.

Essentially, a tolerance relation R defined on a set X decomposes the set into its several subclasses, henceforth called tolerance classes (TCs). The elements of a TC are pairwise tolerant. A subclass $M \subseteq X$ is called a maximal tolerance class (MTC) if any element of M is tolerant to its every other element and no element of $X - M$ is tolerant to *all* the elements of M. Equivalently, a tolerant class is maximal if it is not a proper subclass of any other tolerant class. Graphically (X finite), MTCs for a given tolerance relation R can also be viewed as the largest *complete* polygons in the graph of R. A polygon in which every node is connected to its every other node is called a *complete* polygon. A triangle is always a complete polygon and, for a quadrilateral to be a complete polygon, we need both diagonals as well. Each complete polygon is a TC, but not necessarily an MTC (since MTCs are only the largest complete polygons). In addition, any element of the set X that relates only to itself is an MTC, and any two elements of X which are tolerant to one another but to no other element of X form an MTC.

The pair $< X, R >$ is called a tolerance space. The points of X can be viewed as nodes in a network system, *states* in a finite machine, *vertices* in a graph, or *m-dimensional* vectors in a data structure. A family of TCs covering X will be called a *cover* of X. A family M of all MTCs, which is clearly a cover of X, will be called a *complete cover* of X, and if no proper part of M is a complete cover of X then M will be called a *minimal* complete cover of X. A minimal family of MTCs covering X is called a *basis* for $< X, R >$. Moreover, since a tolerance relation, unlike an equivalence relation, is not necessarily transitive, some MTCs of a complete cover may not be disjoint. If all the MTCs of a complete cover are disjoint, then it is a partition. A wide variety of algorithms to compute MTCs of a tolerance relation (defined on a finite set) are known to exist [2, 4, 11, 13, 21, 22, 24, 28, 34, 35, 37, 38]. It may, however, be noted that besides a heuristic method, no formal algorithmic approach to compute bases of a tolerance space is known. Interestingly, algorithms for determining equivalence and tolerance classes that exploit some forms of their vector representations have also been introduced [2, 11, 13, 22, 23, 28, 35, 37, 38].

Tolerance relations are found useful in solving a class of minimization or minimalization problems, particularly when the problems are incompletely specified: switching theory, sequential machines, design of digital control units, combinatorial problems (such as the scheduling of traffic control), centralization versus decentralization in networks and information systems, phonology, etc., [4, 11, 14, 24, 28, 29, 32, 35, 37, 42]. One of the most important applications of minimal complete covers is found in eliminating redundancies occurring in networks and information systems [2-4, 11, 14, 24, 28, 32, 33, 38, 39, 42].

Owing to the introduction of fuzzy sets and fuzzy relations as a generalization of crisp sets and crisp relations [40, 41], most of the features describing various concepts in the crisp case (briefly described above) have been conveniently extended to describe similar concepts in the fuzzy framework. In particular, the profound mathematics of crisp equivalences and tolerance relations has been exploited to develop the mathematics of fuzzy equivalence and fuzzy tolerance relations. It is well known that studies directed to characterization and modelling of fuzzy equivalence and fuzzy tolerance relations have attracted wide attention [1, 3, 5-8, 10, 12, 16-18, 25-27, 29, 30, 32, 36, 39, 42, 43]. A recently published survey [31] contains almost all advancements and corresponding references on this topic. Nevertheless, the concept of a vector-approach to represent fuzzy equivalence and fuzzy tolerance classes has not been explicitly explicated.

This paper, besides the basics of fuzzy equivalence and tolerance relations, primarily aims at describing methods for vector representation of equivalence classes of an α-partition and tolerance classes of an α-cover. In addition, a method for the derivation of fuzzy partitions and fuzzy covers from a given vector structure is developed. Finally, certain monoids of α-partitions and α-covers are introduced. Throughout the paper, we shall confine our discussions to [0,1]-fuzzy relations. At the outset, it should be noted that instead of the terminologies TCs and MTCs, tolerant pre-classes and tolerant classes have respectively been in use for quite some time in most of the papers.

2. SOME BASIC CONCEPTS

We include some basic concepts, largely drawn from [3, 10, 17, 18, 20, 32, 39], to make this paper minimally self-contained.

Definition 1: Fuzzy Relations

A binary fuzzy relation $\underline{R}$ from a universe set X to an another universe set Y, denoted $\underline{R}(X,Y)$ is a fuzzy set in $X \times Y$, characterized by the membership function

$\mu_{\underline{R}} : X \times Y \rightarrow [0,1]$.

In other words,

$\underline{R}(X,Y) = \{((x,y),\ \mu_{\underline{R}}\ (\text{x, y}))|\ \mu_{\underline{R}} : X \times Y \rightarrow [0,1], x \in X,\ y \in Y\}$.

We say, $(x,y) \in \underline{R}(X,Y)$ iff $0 \leq \mu_{\underline{R}}\ (x\ ,y) \leq 1,\ x \in X\ ,\ y \in Y$; that is, $\mu_{\underline{R}}\ (x\ ,y) = \alpha$ iff $(x,y) \in \underline{R}(X,Y)$ with the degree of membership $\alpha \in [0,1]$.

If $X = Y$, then $\underline{R}(X,X)$ is said to be a fuzzy binary relation in a single universe set X, expressed by the membership function $\mu_{\underline{R}}: X \times X \rightarrow [0,1]$.

Similarly, an n-array fuzzy relation can be defined.

A fuzzy relation, in general, may be viewed as a kind of fuzzy set.

Similar to representing crisp binary relations, fuzzy binary relations (defined on a finite set) can be represented by using sagittal diagrams, matrix forms, and graphical forms. In these representations, $\mu_{\underline{R}}(x, y)$, the membership grade or degree of (x, y) in $\underline{R}$, is represented by a line, a term, and an edge, respectively. It is viewed as indicating the strength of the relation $\underline{R}$ between x and y, that is, the higher $\mu_{\underline{R}}(x, y)$ signifies the higher *bonding* between x and y. For instance, $\mu_{\underline{R}}(x, y) = 1$ indicates that x and y are *most strongly* related. Similarly, if $\mu_{\underline{R}}(x_1, y_1) \leq \mu_{\underline{R}}(x_2, y_2)$ then (x_1, y_1) is said to be *less strongly* related than (x_2, y_2). Moreover, as every member of the class $\{\mu_{\underline{R}}(x_i, y_j);\ i, j = 1,2, \ldots\}$ is a real number, a linear ordering exists between any two members.

It is well-recognized that the aforesaid interpretation of $\mu_{\underline{R}}(x, y)$ plays an essential role in comprehending the *core* of a fuzzy relation, particularly in its application to knowledge representation. It provides a means to achieve the *substitutivity* property of equality in a fuzzy framework. For instance, the linguistic statement *x is more strongly related to y than z* can be objectified as $\mu_{\underline{R}}(x, y) \geq \mu_{\underline{R}}(x, z)$. This is how fuzzy relations, compared to crisp relations, acquire much higher expressive power.

In the following, we shall be mainly concerned with fuzzy binary relations in a single (finite) universe set X with the valuation set $[0,1]$. Moreover, the modifier binary, valuation set $[0,1]$, and the universe set X may not be explicitly mentioned.

Definition 2: α-cuts of fuzzy relations

Just as an α-cut of a fuzzy set is a crisp set containing all the elements of the universe set X having a membership grade not less than α, an α-cut of a fuzzy relation $\underline{R}$ is an α-cut (crisp) relation $\underline{R}_{\alpha}$, defined by

$$\underline{R}_{\alpha} = \{(x, y) |\ \mu_{\underline{R}}(x, y) \geq \alpha,\ (x, y) \in X \times X, \alpha \in \wedge_{\underline{R}}, \textit{the level set of } \underline{R}\},$$

and its membership function by

$\mu_{\underline{R}_{\alpha}}(x, y) = 1$, whenever $\mu_{\underline{R}}(x, y) \geq \alpha$, and equals 0 otherwise.

It may be emphasized that $\mu_{\underline{R}}(x, y) \geq \alpha$ indicates that x and y are $\underline{R}$-related to a degree not less than α.

The method of α-cuts was originally introduced by Zadeh [40] and subsequently studied by a number of researchers ([10, 16, 17, 18, 20, 32], to mention a few). It has been found a powerful tool for *defuzzification.*

Definition 3: Properties of Fuzzy Relations

Let $\underline{R}$ be a fuzzy relation in X.

$\underline{R}$ is said to be

- *reflexive* iff $(\forall\ x \in X)\ (\mu_{\underline{R}}(x,x) = 1)$,

  (sometimes called *strict reflexivity* to distinguish it from other kinds of reflexivity(see [5, 12, 39] and especially [31] for details))

- *symmetric* iff $\forall\ (x,y) \in X \times X, \mu_{\underline{R}}\ (x,y) = \mu_{\underline{R}}\ (y,x)$;
- *transitive* iff $\forall\ (x,z) \in X \times X$,
  $\mu_{\underline{R}}(x,z) \geq \max_{y}\ (\min(\ \mu_{\underline{R}}\ (x,y),\ \mu_{\underline{R}}\ (y,z)),$

  (Sometimes called *max-min* transitivity to distinguish it from other kinds of transitivity (related references are cited at the end of the introduction)), and

- *not transitive* iff $\exists\ (x,z) \in X \times X$,
  $\mu_{\underline{R}}(x,z)\ <\ \max_{y}\ (\min(\ \mu_{\underline{R}}\ (x,y),\ \mu_{\underline{R}}\ (y,z)).$

  Essentially, the aforesaid definitions of various properties are the fuzzified versions of their crisp analogs.

Definition 4: Fuzzy Equivalence (or Similarity) Relation

A fuzzy relation $\underline{R}$ in X is said to be a fuzzy *equivalence* relation (aka *indistinguishability* or *similarity* relation) on X if it is reflexive, symmetric, and transitive. If $\underline{R}$ is a similarity relation, $\mu_{\underline{R}}\ (x,y)$ denotes the degree or value of similarity between x and y. In [30], the following characterization of the similarity measure has been developed: $\underline{R}$ is a similarity relation iff for any three similarity values α, β, γ; either $\alpha = \beta = \gamma$, or two of them are equal and the third is larger. Equivalently, $\beta \wedge \gamma = \alpha \wedge \beta \wedge \gamma, \alpha \wedge \beta = \alpha \wedge \beta \wedge \gamma, \alpha \wedge \gamma = \alpha \wedge \beta \wedge \gamma$ ([27] contains related details).

Let $\underline{R}$ be a fuzzy equivalence relation. If an α-cut is applied to $\underline{R}$, we obtain an α-cut (crisp) equivalence relation $\underline{R}_\alpha$ expressed as in Definition 2. In this way, a fuzzy equivalence relation can be viewed as a fuzzified version of an ordinary equivalence relation. Each $\underline{R}_\alpha$, $\alpha \in \wedge_{\underline{R}}$, the level set of $\underline{R}$, defines an α-cut partition $\pi(\underline{R}_\alpha)$. The class of all α-cut partitions $\{\pi(\underline{R}_{\alpha_i}), \alpha_i \in \wedge_{\underline{R}}\}$, denoted by $\Pi(\underline{R})$, is a *nested* family in the sense that $\pi(\underline{R}_{\alpha_2})$ is a *refinement* of $\pi(\underline{R}_{\alpha_1})$ whenever $\alpha_1 \leq \alpha_2$. As is known, refinement in general is different from set containment, that is, $\pi(\underline{R}_{\alpha_2})$ being a refinement of $\pi(\underline{R}_{\alpha_1})$ does not necessarily imply $(\underline{R}_{\alpha_2}) \subseteq \pi(\underline{R}_{\alpha_1})$.

It may be observed that, like a crisp equivalence relation, a fuzzy equivalence relation $\underline{R}$ on X generates a unique partition of X for each $\alpha \in \wedge_{\underline{R}}$. Moreover, for any $\alpha_1, \alpha_2 \in \wedge_{\underline{R}}$ with $\alpha_1 \leq \alpha_2$, no two elements of X can be in the same α_2 equivalence block if they were not already in the same α_1-equivalence block. We consider the following example to illustrate the aforesaid descriptions and for further deliberations.

Example 1

Let $\underline{R}$ be a fuzzy equivalence relation (≈) on $X = \{1,2,3, \dots, 7\}$, defined by the following membership matrix (Table 1), followed by its simplified undirected weighted graph (Figure 1), and the sequence of all α-partitions (List 1):

Table 1. Membership Matrix for $\underline{R}$ on X

≈	1	2	3	4	5	6	7
1	1.0	0.0	0.0	1.0	0.7	0.6	0.0
2	0.0	1.0	0.9	0.0	0.0	0.0	0.0
3	0.0	0.9	1.0	0.0	0.0	0.0	0.0
4	1.0	0.0	0.0	1.0	0.7	0.6	0.0
5	0.7	0.0	0.0	0.7	1.0	0.6	0.0
6	0.6	0.0	0.0	0.6	0.6	1.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	1.0

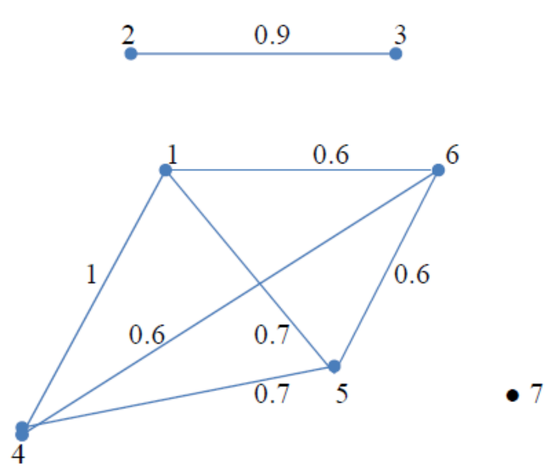

Figure 1. Simplified undirected weighted graph for $\underline{R}$

Since a similarity relation is reflexive and symmetric, in its simplified graph, neither loops at each node nor directions between every two nodes need to be drawn.

It may be readily seen that the aforesaid matrix and graph do define a similarity relation since they satisfy Potoczny's criteria [30].

For $\wedge_{\underline{R}} = \{0.0, 0.6, 0.7, 0.9, 1.0\}$, the corresponding sequence of all distinct partitions $\{\pi(\underline{R}_{\alpha_i}), \alpha_i \in \wedge_{\underline{R}}\}$, denoted here by $\{P_i, i = 0,1,2,3,4\}$, is given below (List 1):

List 1. Sequence of all α-partitions for $\underline{R}$ on X

$P_0(\alpha = 0.0) = \{\overline{1,2,3,4,5,6,7}\};$

$P_1(\alpha = 0.6) = \{\overline{2,3}, \overline{1,4,5,6}, \overline{7}\};$

$P_2(\alpha = 0.7) = \{\overline{2,3}, \overline{1,4,5},, \overline{6}, \overline{7}\};$

$P_3(\alpha = 0.9) = \{\overline{2,3}, \overline{1,4}, \overline{5}, \overline{6}, \overline{7}\};$

$P_4(\alpha = 1.0) = \{\overline{2}, \overline{3}, \overline{1,4}, \overline{5}, \overline{6}, \overline{7}\}.$

The aforementioned sequential representation of all α-partitions for a fuzzy equivalence relation can also be represented by a *tree* diagram (e.g. [17, 18, 20, 27, 32]), called a *partition tree* or *hierarchical tree*. The notion of a partition tree can be viewed as a generalization of a *quotient set* defined in the classical theory of equivalence relations ([27] contains further details). In fact, every fuzzy equivalence relation defined on a finite set X generates an indexed hierarchical tree on X ([31] is a very recently published compendium of most of the advancements that have taken place concerning fuzzy equivalence relations, in particular).

Definition 5: Fuzzy Tolerance Relation

A fuzzy relation $\underline{R}$ in X is said to be a fuzzy *tolerance* relation (aka *proximity* or *resemblance* relation) if it is reflexive and symmetric. If an α-cut is performed on a fuzzy tolerance relation $\underline{R}$, we obtain an α-cut (crisp) tolerance relation $\underline{R}_\alpha$, expressed as in Definition 2. In this way, a fuzzy tolerance relation can be viewed as a fuzzified version of an ordinary tolerance relation.

Let $\underline{R}$ be a fuzzy tolerance relation on X. An α-tolerance class is a subset of X in which elements are pairwise tolerant to a degree not less than α. For each $\alpha \in \wedge_{\underline{R}}$, the level set of $\underline{R}$, the collection of all corresponding α-tolerance classes constitute a complete α-cover of X, denoted $\pi(\underline{R}_\alpha)$, instead of a partition of X since $\underline{R}$ is not necessarily transitive. It may, however, be noted that for some values of α, the corresponding α-covers may form partitions of X.

Similar to fuzzy equivalence relations, the collection of all complete α-covers forms a *nested* family. For $\alpha_1, \alpha_2 \in \wedge_{\underline{R}}$ with $\alpha_1 \leq \alpha_2$, no two elements of X can be in the same α_2-tolerance class if they were not already in the same α_1-tolerance class. We consider the following example to illustrate the aforesaid descriptions and for further deliberation.

Example 2

Let $\underline{R}$ be a fuzzy tolerance relation (~) on $X = \{1,2, \dots, 7\}$ defined by the following membership matrix (Table 2), followed by its simplified undirected weighted tolerance graph (Figure 2), and the sequence of all complete α-covers (List. 3):

Table 2. Membership Matrix for $\underline{R}$ on X

~	1	2	3	4	5	6	7
1	1.0	0.0	0.0	1.0	0.7	0.0	0.0
2	0.0	1.0	0.9	0.0	0.0	0.0	0.0
3	0.0	0.9	1.0	0.0	0.0	0.0	0.0
4	1.0	0.0	0.0	1.0	0.7	0.6	0.0
5	0.7	0.0	0.0	0.7	1.0	0.6	0.0
6	0.0	0.0	0.0	0.6	0.6	1.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	1.0

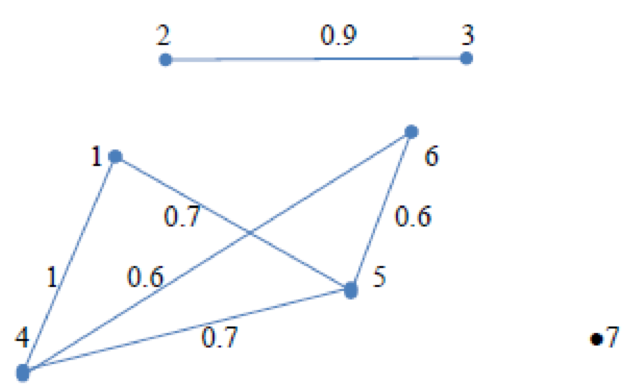

Figure 2. Simplified tolerance Graph for $\underline{R}$

Since $\mu_{\underline{R}}(1,5) = 0.7$, $\mu_{\underline{R}}(5,6) = 0.6$, but $\mu_{\underline{R}}(1,6) = 0 < \min(0.6, 0.7)$, $\underline{R}$ is not min-transitive. It is immediate to see that none of the other forms of transitivity holds as well. It may be an interesting exercise to construct examples of tolerance relations on a finite set X that are non-transitive for one form of transitivity but transitive for some others (see [1]).

For $\wedge_{\underline{R}} = \{0.0, 0.6, 0.7, 0.9, 1.0\}$, the corresponding sequence of all complete α-covers $\{C_i, i = 0,1,2,3,4\}$ is given below (List 2):

List 2. Sequence of all complete α-covers for $\underline{R}$ on X

$C_0(\alpha = 0.0) = \{\overline{1,2,3,4,5,6,7}\}$;

$C_1(\alpha = 0.6) = \{\overline{2,3}, \overline{1,4,5}, \overline{4,5,6}, \overline{7}\}$;

$C_2(\alpha = 0.7) = \left\{\overline{\overline{2,3}, \overline{1,4,5}, \overline{6}, \overline{7}}\right\}$;

$C_3(\alpha = 0.9) = \{\overline{2,3}, \overline{1,4}, \overline{5}, \overline{6}, \overline{7}\}$;

$C_4(\alpha = 1.0) = \{\overline{2}, \overline{3}, \overline{1,4}, \overline{5}, \overline{6}, \overline{7}\}$.

All complete α-covers for $\alpha \geq 0.7$ are partitions as well.

The aforementioned sequential representation of all α-covers can also be described as a *tree* diagram, sometimes called a *cover tree*. It may be noted that each α-tolerance class of an α-cover at every level $\alpha \in \wedge_{\underline{R}}$ is maximal in the sense that none of these classes is a proper part of the other.

Fuzzy tolerance relations and the analysis of fuzzy tolerance spaces have been studied and applied by a number of researchers [1, 3, 6, 7, 15, 23, 29, 33, 36, 42]. A deep study into the structural properties of tolerance spaces can be found in [1, 7]. In particular, defining tolerance through an application of concept lattices in [1] is a novel approach. In [31], many related results and references can be found.

3. VECTOR REPRESENTATION OF α-PARTITIONS AND α-COVERS

In order to develop algorithms for determining crisp equivalence and tolerance classes, the notions of FIRST and MEMBER vectors [37] have been employed. The procedure is outlined as follows:

Let the universe set X be an n-set. Accordingly, both FIRST and MEMBER vectors comprise n components. The kth component of *FIRST*, $1 \leq k \leq n$, contains the number which is the first element in the maximal tolerance class(es) to which k belongs. The kth component of *MEMBER* contains the number(s) following k. In the case of k being the last element, *MEMBER* $[k] = 0$. It may also be noted that if k itself is the first element, *FIRST* $[k] = k$. Since α-equivalence classes as well as α-tolerance classes are crisp classes, in order to define their vector representations, the method described above applies. We illustrate it as follows:

3. 1. Vector representation of fuzzy equivalence classes (Example 1, List. 3)

Let $\alpha = 0.6$. The crisp classes of the corresponding α-partition are:

$E_1 = \{2,3\}, E_2 = \{1,4,5,6\}, and\ E_3 = \{7\}$.

FIRST	Elements	MEMBER
{1}	1	{4}
{2}	2	{3}
{2}	3	{0}
{1}	4	{5}
{1}	5	{6}
{1}	6	{0}
{7}	7	{0}

Elucidation 1: Vector Representation of α-partition for $\alpha = 0.6$

3. 2. Vector representation of fuzzy tolerance classes (Example 2, List 2)

Let $\alpha = 0.6$. The crisp classes of the corresponding α-cover are

$C_1 = \{2,3\}, C_2 = \{1,4,5\}, C_3 = \{4,5,6\}, and\ C_4 = \{7\}$.

FIRST	Elements	MEMBER
{1}	1	{4}
{2}	2	{3}
{2}	3	{0}
{1,4}	4	{5,5}
{1,4}	5	{0,6}
{4}	6	{0}
{7}	7	{0}

Elucidation 2: Vector Representation of α-cover for α = 0.6

4. DERIVATION OF α-PARTITIONS AND α-COVERS FROM A GIVEN VECTOR STRUCTURE

Here, we describe a method to derive α-partitions and α-covers from a given vector structure. Let $1 \leq i, j, k, \ldots \leq n$ where n is the number of elements in X.

Step 1: The number of equivalence/tolerance classes in an α-partition/α-cover equals the number of zeros (repetitions counted, if any) in MEMBER. Effectively, this step provides an indicator of the completion of the task.

Step 2: List $\mathrm{i, j, k, \ldots}$ in the same class if the sets of their components form a $\subseteq$-chain (subset hood-chain, including equality), that is,

$\mathrm{FIRST}\ [i] \subseteq \mathrm{FIRST}\ [j] \subseteq \ldots$

Additionally, if $\mathrm{FIRST}[i] = \mathrm{FIRST}[j] = \ldots$, and the cardinality of their component set is m, then $i, j, \ldots$ appear together in m number of classes. This information helps verify the results obtained.

Step 3: $(\forall\ i, j)$ $\mathrm{FIRST}[i] \cap \mathrm{FIRST}[j] \neq \emptyset\ \Rightarrow i$ and j belong to the same class.

Step 4: List $\mathrm{i, j, k}$ in the same class if $\mathrm{(i, j), (j, k)}$, and $\mathrm{(i, k)}$ hold together, for all $\mathrm{i, j, k}$ as described in step 3.

Moreover, to avoid repetitions, the pairs $\mathrm{(i, j)}$ for all $\mathrm{i, j}$ need not be considered if they have already been treated in the previous steps.

It may be observed that the results obtained in steps 3 and 4 would eventually include the results obtained in step 2. Nevertheless, if the $\subseteq$-chain for a given problem involves three or more terms, step 2 will be found extremely helpful.

We illustrate the aforesaid procedure as follows:

For brevity, let (FE, ME) and $(FT\ , MT)$ stand for $(FIRST, MEMBER)$ indicating equivalence and tolerance, respectively.

4. 1. Derivation of α-equivalence classes (Example 1, Elucidation 1)

Let $\alpha = 0.6$

Step 1: Since there are three zeroes in ME, there will be only three equivalence classes in the α-partition for $\alpha = 0.6$.

Step 2: $\text{FE}[1] = \text{FE}[4] = \text{FE}[5] = \text{FE}[6] \Rightarrow \{1,4,5,6\} = E_2$

$\text{FE}[2] = \text{FE}[3] = \{2,3\} = E_1$

$\text{FE}[7] = \text{FE}[7] = \{7\} = E_3$

The procedure terminates since all the classes are obtained [Step 1].

4. 2. Derivation of α-tolerance classes (Example 2, Elucidation 2)

Let $\alpha = 0.6$

Step 1: Since there are four zeroes in MC, there will be only four maximal-compatibility classes in the α-cover for $\alpha = 0.6$.

Step 2: FT [1] ⊆ FT [4] = FT [5] ⇒ {1,4,5} $= C_2$

$\text{FT}\ [6] \subseteq \text{FT}\ [5] = \text{FT}\ [4] \Rightarrow \{4,5,6\} = C_3$

$\text{FT}\ [2] = \text{FT}\ [3] \Rightarrow \{2,3\} = C_1$

$\text{FT}\ [7] = \text{F}[7] \Rightarrow \{7\} = C_4$

The procedure terminates since all the classes are obtained [Step 1].

It may be observed that $\text{FT}\ [4] = \text{FT}\ [5] = 2 \Rightarrow 4$ and 5 will appear together in two distinct classes, and $\text{FT}\ [2] = \text{FT}\ [3] = 1 \Rightarrow 2$ and 3 will appear together in only one class.

We note at this end that algorithms for computing fuzzy equivalence and tolerance classes are developed in [1, 3, 5-8, 10, 15, 17, 18, 20, 32, 33, 39, 42].

5. MONOIDS OF α-PARTITIONS AND α-COVERS

Besides a wide variety of applications of monoids in syntactic analysis and formal language theory, monoids of partitions are found particularly useful in the study of sequential machines [22, 23, 28, 37]. It is a very useful application of the property possessed by α-partitions of a given set X for a given fuzzy equivalence relation $\underline{R}$ on X that they form a monoid under some suitable operations defined on them.

Let $\otimes$ be a binary operation on the class $\Pi(\text{X})$ consisting of all α-partitions of X. Let $P = \{P_1, P_2, \ldots\}$ and $Q = \{Q_1, Q_2, \ldots\}$ be any two α-partitions of X. We define $P \otimes Q$ as the class of non-empty intersections of each class of P with each class of Q. It can be easily seen that the operation $\otimes$ is both commutative and associative. Moreover, the partition $\{\overline{X}\}$ for $\alpha = 0.0$ is the *identity* of the operation $\otimes$, and every member of the class of all α-partitions of X is idempotent with respect to the operation $\otimes$. The procedure can easily be verified by taking the aforementioned Example 1 into consideration. It may be noted that this approach has a precedent in the theory of partitions for sets [37], and usually, it is called the "product of the partitions".

Let $\oplus$ be another binary operation defined on $\Pi(X)$ as follows:

For $G, H \in \Pi(X)$, a subset Y of X belongs to $G \oplus H$, if

i. Y is the union of one or more elements of G,
ii. Y is the union of one or more elements of H, and
iii. No subset of Y satisfies (i) and (ii) except Y itself.

The operation $\oplus$ is commutative and associative. Also, the α-partition ($\alpha = 1$) consisting of single elements of X is the identity for the operation $\oplus$ defined on $\Pi(X)$.

Similarly, monoids of all α-covers of X can be described. However, unlike an α-partition, tolerance classes in a complete α-cover may not be pairwise disjoint. Consequently, on applying the operations as defined in the case of α-partitions, the resulting α-cover may not be a complete α-cover in the sense that some of its tolerance classes may not be maximal. Nevertheless, it will always define a cover for some $\alpha \in [0,1]$. Thus, the closure property is satisfied.

For instance, considering the aforementioned Example 2, we have,

$C_1 \otimes C_2 = \{\overline{2,3}, \overline{1,4,5}, \overline{4,5}, \overline{6}, \overline{7}, \}$, in which $\overline{4,5}$ is a proper part of $\overline{1,4,5}$, and thus it is not a complete α-cover consisting of maximal tolerance classes alone. However, by deleting such classes along with their repetitions (if any), a minimal complete α-cover can be obtained. Moreover, idempotency with respect to none of the operations defined above holds.

6. CONCLUDING REMARKS

In this paper, a method to construct vector structures for representing maximal tolerance classes has been described and extended to constructing vector structures for fuzzy equivalence and fuzzy tolerance classes. A technique for deriving fuzzy equivalence and tolerance classes from a given vector structure has been introduced. In addition, certain monoids of α-partitions and α-covers have been defined.

It is envisaged that the methods described in this paper may be found useful in developing efficient algorithms to obtain fuzzy partitions and complete covers. Moreover, defining some other suitable operations which would give rise to certain monoids of fuzzy partitions and fuzzy covers (particularly the ones in which idempotency holds) may be a challenging and interesting problem.

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