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Controllability of semilinear systems using the banach contraction mapping principles

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ABSTRACT

In this work, we studied the controllability of Semilinear Systems in Banach Spaces. By using the Banach's contraction mapping principle, the Semilinear system was reduced into a Linear system whose solution by Picard's iteration method converges to a unique fixed point. By employing the Kalman's controllability Criterion, the system was transformed into a controllability Gramian with rank n. This established the controllability of the system.

***Keywords*:** Controllability, Semilinear systems, Fixed point, Rank of a Matrix, Banach Contraction Mapping, Lipschitzian and Stability

1. INTRODUCTION

Controllability is one of the basic ideas in Mathematical systems theory. It is an important property of dynamical control systems and it plays a vital role in control theory. Recently, the

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theory of control of deterministic processes with some degrees of freedom have reached a satisfying stage of completeness.

This was interpreted by the work of [4]. The fundamental problems of control theory have been mathematically posed and answered and hence the theory has reached a certain degree of stability and perfection. With the current status of the theory, the authors believed that a careful presentation of the theory would serve the purpose of offering a foundation on which further researches could be based. The intention of this research is to establish the controllability of semilinear differential equation using Banach contraction principle. This is to present a feasible conditions on controllability of semilinear systems that could be thorough and complete within the limitations set by restriction of deterministic problems which results in an ordinary differential system.

The origin of controllability as a research interest can be traced to series of investigations initiated by [9]. [9] used the centrifugal governor (a specific type of governor with a feed back system that controls the speed of an engine) to control his steam engine where it regulates the admission of steam into the cylinder(s). Other significant works in the early stages of development of control are done by authors like Minorsky, Hazen, and Nyquist, among many others.

[13] developed an automatic controllers for steering ships and showed how stability could be determined from the differential equations describing the system.

[18] worked on a relatively simple procedure for determining the stability of closed-loop systems on the basis of open-loop response to steady-state sinusoidal inputs.

[11] developed sufficient conditions for the null controllability of a non finite delay system with implicit derivatives.

[8] discovered the term servomechanisms for position control systems, discussed the design of relay servomechanisms capable of closely following a changing input. Over the intervening years, the idea of controllability of nonlinear systems described by ordinary differential equation in finite and infinite dimensional spaces has been studied by several authors who made great advances to ensure that the subject matter grow in leaps and bounds to attract contributions from other international scholars.

[19] dealt with stability theory of inequalities. In his work tittled „differentiable nonlinear systems part two". He described a condition for stability of the solution set of a system of nonlinear inequalities over a closed convex set in a Banach space, when the functions defining the inequalities are subjected to small perturbations. According to him, the condition involves the linearization of the system about a point and it is shown to be sufficient and, under a weak additional hypothesis, also necessary for stability. Quantitative estimates for the changes in the solution set are obtained.

[15] worked on controllability of semilinear control systems dominated by the linear part. According to Naito, while various equivalent conditions for controllability have been obtained in the case of linear control systems, controllability problems of semilinear control systems usually require some complicated and limited assumptions. Naito showed the approximate controllability of an abstract semilinear control system under the assumption, which has a simple form and can be easily checked in many examples.

[21] investigated the existence and properties of a class of partial functional and differential equations. [4] worked on controllability of nonlinear systems via fixed point theorems. In their work, they transformed the controllability problem to a fixed-point problem

for an appropriate nonlinear operator in a function space. According to them an essential part of this approach is to guarantee the existence of an invariant subset for this operator.

[5] studied the controllability of Integro-differential Systems in Banach spaces, also with delay term. In their work, they used Schaefer fixed-point theorem to establish sufficient conditions for controllability of semilinear functional resulting to differential systems in a Banach space.

[3] established sufficient conditions for the controllability of quasilinear delay integro-differential systems in Banach spaces and their results were obtained by using the theory of semi group of operators and shauder- tychonov theorem.

In 2007, [6] proved the exact controllability of nonlinear third order dispersion equation and obtained controllability results using two standard types of nonlinearities, namely, Lipschitzian and monotone. According to them it is possible to obtain the exact controllability of the same system through the approach of Integral Contractors which is a weaker condition than Lipschitz condition. In 2009, [7] worked on controllability of semilinear differential systems with local conditions in Banach spaces.

[20] showed that the existence theory on controllability in literature can trivially be adjusted to include the infinite dimensional spaces setting if we replace the compactness of the operators with the complete continuity of the nonlinearity.

[14] worked on fractional integro-differential systems in Banach spaces.

The work of [12] shows the controllability of semi –linear differential equations and inclusion via semi group theory in banach spaces. In 2013, [10] reviewed the major progress that has been made on controllability of dynamical systems over the past number of years and finally, it is well known, that controllability concept has many important applications not only in control theory and systems theory, but also in such areas as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospce engineering and recently in quantum systems theory.

[1] proved the existence of solutions for three-point functional differential inclusions with P-Laplacian operator. In 2016, [2] obtained a sufficient conditions for approximate controllability of various tpes of dynamic systems using Shauder-fixed point theorem.

[17] established conditions that guarantees a new type of controllability of Semilinear systems in Banach space and he obtained his result by using Shauder fixed point theorem and in the same year, [16] establishes necessary and sufficient conditions for exponential asymptotic stability in the large and uniform asymptotic stability of perturbations of linear delay systems and according to him, if the exponential estimate for the solution of a system tends to zero as time t tends to infinity, the system is said to be uniformly asymptotically stable.

[22] determined the controllability for semilinear functional differential equations without uniqueness using the fixed point theory for multivalued maps with non-convex values. Here in this research work, we determined the controllability of the same semilinear differential systems and obtain existence of result and uniqueness using Banach Contraction Principle and Picard's iteration method. The result confirms that a unique solution of semilinear differential systems can be obtained and the system can be controlled using Banach Contraction Principle.

2. METHODOLOGY

We consider the following semilinear system

$$\left.\begin{aligned} &x'(t) = Ax(t) + Bu(t) + F(t, x(t), x_t); t \in J := [a,b] \\ &x(s) = \phi(s); s \in [a,b] \\ &x(t_o) = x_0 \end{aligned}\right\} \tag{2.1}$$

where the state space $x(t)$ takes values in a Banach space X. A is a bounded linear operator in $X, u(t) \in V$ is the admissible control that belongs to the Banach space V. Bis any map on V and $F: J \times X \times C([a,b], X) \rightarrow X$ is a non-linear operator which is continuous in the first variable and Lipschitzianin the second variable. The history $x_t$is defined by $x_t(s) := x(t+s)$ for $s \in [a,b]$.

In dealing with the control system (2.1), we employ the method of Banach contraction mapping theorem. This method enables us to show that there is a contraction on X which has a unique fixed point. We also used Picard's iteration method to show the existence and uniqueness of the solution. Furthermore, we use the Kalman's controllability theorem to show that the system is controllable.

Theorem 2.1. (Banach Contraction Mapping Principle)

Let X be a Banach space. If $T: X \rightarrow X$ is a contraction, then T has a unique fixed point say x^*. That is,

$$Tx^* = x^* \tag{2.2}$$

Furthermore, for fixed $x_0 \in X$ and given the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = Tx_{n-1}$where $n = 1, 2, 3, \ldots$, then the sequence converges to a unique fixed point x^* of T.

Theorem 2.2. (Picard's iteration theorem)

Given an initial value problem (IVP)

$$\frac{dy}{dt} = f(x,y), y(x_0) = y_0 \tag{2.3}$$

$$f : D \subseteq R^2 \rightarrow R$$
$$(x_0, y_0) \in D$$

Let D be a domain in R^2 and $f : D \rightarrow R$ be real function satisfying the following conditions:

(i) f is continuous on D

(ii) $f(x,y)$is Lipschitz continuous with respect to y on D with lipschitz constant $\alpha > 0$.

Let (x_0, y_0) be interior point on D and let $a > 0, b > 0$ be constants such that the rectangle

$\{(x,y):|x-x_0|\leq a, |y-y_0|\leq b\}\subset D$,

Let,

$$M=\max_{(x,y)\in R} f(x,y), h=\min\left(a, \frac{b}{m}\right)$$

Then, the IVP has a unique solution y on the interval $|x-x_0|\leq h$.

2. 1. Method of Controllability

The question is "Can a control function $u(t)$ be found which will transform the initial state x_0 of a system to some desired final state x_1 in finite time"?

Consider the linear part of equation (2.1) which described the state equation

$$x = Ax + Bu \qquad 2.4$$
$$y = Cx$$

where $A, B,$ and C are matrices of sizes $n\times n, n\times m, r\times n$ respectively.

We define the transformation $x = Pz$ where P is a non – singular $n\times n$ matrix. Then, we have that;

$$z = A_1 + B_1 \qquad 2.5$$
$$y = c_1 z$$

where $A_1 = P^{-1}AP,\ B_1 = P^{-1}B, C_1 = CP$.

If A has distinct eigenvalues, then A_1 is a diagonalized matrix with the eigenvalues as the entries at the diagonal of the matrix. Thus,

$$A_1=\begin{pmatrix}\lambda_1 & 0 & 0 & \dots & 0\\ 0 & \lambda_2 & 0 & \dots & 0\\ . & . & . & \dots & .\\ 0 & 0 & 0 & \dots & \lambda_n\end{pmatrix}$$

Without loss of generality, If $m = n = r = 2$, then

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}+\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

$$\dot{z_1} = \lambda_1 z_1 + b_{11}u_1 + b_{12}u_2$$

$$\dot{z_2} = \lambda_2 z_2 + b_{21}u_1 + b_{22}u_2$$

$$y_1 = C_{11}z_1 + C_{12}z_2$$

$$y_2 = C_{21}z_1 + C_{22}z_2$$

We can write

$$\dot{z}_1 = \lambda_1 z_1 + b_1^T u$$
$$z_2 = \lambda_2 \dot{z}_2 + b_2^T u$$

where $b_1^T = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}$ and $y = c_1 z_1 + c_2 z_2$ where $c_1 = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}$ and $c_2 = \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}$

In general (2.5) can be written in the form

$$z_i = \lambda_i z_i + b_i^T (i = 1,2, \dots, n)$$
$$y = \sum_{i=1}^{n} c_i z_i \qquad 2.6$$

It can be seen that from (3.6) if b_i^T (i^{th} row of B) has all zero components, then $\dot{z}_i = \lambda_i z_i$ and the input $u(t)$ has no influence on the i^{th} mode of the system. The mode is said to be uncontrollable, and the system having one or more such modes is uncontrollable. Otherwise, where all the modes are controllable, the system is said to controllable.

Theorem 2.3. (Kalman's theorem)

Kalman's controllability theorem states that if the rank of the matrix Q is n (Q is nonsingular), the system is controllable if the rank is less than$n$ (Q is singular) and the system is uncontrollable.

The matrix Q is called the system controllability matrix or Grammian. For multivariable system

$$Q = \left[B \middle| AB \middle| \dots \middle| A^{n-1} B \right] \qquad 2.7$$

Theorm 2.4. let X be a Banach space $J = [a,b] \subseteq \mathbb{R}^+, V = J \times X \times C$ and $C \equiv C([a,b], X) \coloneqq \{T: [a,b] \to X: T \text{ } is \text{ } continuous\}$. Consider the semilinear system as described in (3.1).

The system has a unique solution if

$$(b-a)[||A|| + L] < 1 \qquad 2.8$$

Proof:

Observe that system (3.1) is equivalent to

$$x(t) = \int_a^b Ax(t)dt + \int_a^b Bu(t) + \int_a^b F(t, x(t), x_t)dt$$

set,

$$Tx(t)=\int_a^b Ax(t)\,dt+\int_a^b Bu(t)\,dt+\int_a^b F\left(t,x(t),x_t\right)dt$$

This implies that

$$Ty(t)=\int_a^b Ay(t)\,dt+\int_a^b Bu(t)\,dt+\int_a^b F\left(t,y(t),y_t\right)dt$$

Now,

$$\|Tx-Ty\|=\left\|\int_a^b A\big(x(t)-y(t)\big)dt+\int_a^b F\left(t,x(t),x_t\right)-F\left(t,y(t),y_t\right)\right\|$$

$$\leq\left\|\int_a^b A\big(x(t)-y(t)\big)dt\right\|+\left\|\int_a^b [F\left(t,x(t),x_t\right)-F\left(t,y(t),y_t\right)]dt\right\|$$

$$\leq\int_a^b\left\|A\big(x(t)-y(t)\big)dt\right\|+\int_a^b\left\|[F\left(t,x(t),x_t\right)-F\left(t,y(t),y_t\right)]dt\right\|$$

$$\leq\int_a^a\|A\|\|x(t)-y(t)\|+\int_a^b L\|x(t)-y(t)\|$$

$$=\|A\|\|x(t)-y(t)\|\int_a^b dt+L\|x(t)-y(t)\|\int_a^b dt$$

$$=(b-a)\|A\|\|x(t)-y(t)\|+(b-a)L\|x(t)-y(t)\|$$

$$=(b-a)\big[\|A\|+L\big]\|x(t)-y(t)\|.$$

Using (3.8), we see that T is a contraction on $X$and from the Banach contraction mapping principle, it has a unique fixed point $x^*(t)\in X$ which is the unique solution of the system (3.1).

So, starting from any point $x_0\in X$, the Picard's iteration sequence $x_{n+1}=Tx_n$ converges to the unique fixed point of $x^*(t)$of T.

3. ANALYSIS OF RESULTS/APPLICATIONS

3. 1. Solution of the free system using Picard's iteration method

From (2.1), we have the free system (system without control) as

$$x^{!}(t)=Ax(t),\quad x(t_0)=x_0 \qquad 3.1$$

The Picard's iteration method gives approximate solution to the IVP (3.1). Note that the IVP (4.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_a^b f(t, x(t))dt \qquad 3.2$$

A rough approximation to the solution $x(t)$ is given by the function $x_0(t) = x_0$, which is simply a horizontal line through (t_0, x_0). We insert this to the RHS of (3.2) in order to obtain a (perhaps) better approximate solution, say $x_1(t)$ Thus,

$$x_1(t) = x_0 + \int_a^b f(t, x_0(t))dt = x_0 + \int_a^b f(t, x_0)dt \qquad 3.3$$

The next step is to use this $x_1(t)$ to generate another (perhaps even better) approximate solution $x_2(t)$:

$$x_2(t) = x_0 + \int_{t_0}^{t} f(t, x_1(t))dt \qquad 3.4$$

At the n-th stage we find

$$x_n(t) = x_0 + \int_{t_0}^{t} f(t, x_{n-1}(t))dt \qquad 3.5$$

Convergence guarantees stability hence the solution is stable.

3. 2. Application; Consider the differential system

$$\frac{dx}{dt} = x, \quad x(0) = 1$$

Solution

$$f(t, x) = x \qquad 3.6$$

$$x_0(t) = 1, \quad x_{n+1}(t) = x_0 + \int_0^t x_n(a)da$$

$$x_1(t) = 1 + \int_0^t x_0(a)da = 1 + \int_0^t da = 1 + t$$

$$x_2(t) = 1 + \int_0^t x_1(a)da = 1 + \int_0^t (1+a)da = 1 + t + \frac{t^2}{2} \qquad 3.8$$

$$x_3(t) = 1 + \int_0^t x_2(a)da = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \qquad 3.9$$

$$x_n = \sum_{m=0}^{n} \frac{t^m}{m!} \rightarrow e^t \qquad 3.10$$

3. 3. Kalman's controllability criterion

For single input single out (SISO) B is a column vector, say "b" and C is a row vector "c" such that

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= cx \end{aligned} \qquad 3.11$$

and

$$\begin{aligned} \dot{z} &= A_1 z + b_1 z \\ y &= C_1 z \end{aligned} \qquad 3.12$$

We have established that, the necessary condition for a system defined by (3.11) to be controllable is that the component of the vector $b_1 = [B_1, B_2, B_3 \ldots . B_n]^T$ in (3.12) are all non-zero. It follows that the necessary condition for the (4.11) to be controllable is that the partition matrix

$$Q_1 = \left[b_1 \middle| A_1 b_1 \middle| A_1^{\,2} b_1 \middle| \ldots \middle| A_1^{\,n-1} b_1 \right] \quad =$$

$$\begin{bmatrix} B_1 & \lambda_1 B_1 & \lambda_1^{\,2} b_1 \ldots \lambda_1^{\,n-1} b_1 \\ B_2 & \lambda_2 B_2 & \lambda_2^{\,2} b_2 \ldots \lambda_2^{\,n-1} b_2 \\ . & . & . \quad\quad . \\ . & . & . \quad\quad . \\ . & . & . \quad\quad . \\ B_n & \lambda_n B_n & \lambda_n^{\,n} b_n \quad \lambda_n^{\,n-1} b_n \end{bmatrix} \qquad 3.13$$

is non singular.

Since,

$$A_1 = P^{-1}AP, A_1^{\,2} = P^{-1}A^2P, \ldots, A_1^{\,n-1} = p^{-1}A^{n-1}P$$

$b_1 = P^{-1}b$

therefore,

$$A_1 b = p^{-1}APP^{-1}b = p^{-1}Ab$$
$$A_1^{\ 2}b_1 = p^{-1}A^2 b$$
.
.
.
$$A_1^{\ n-1}b_1 = p^{-1}A^{n-1}b$$

clearly;

$$\Rightarrow Q_1 = \left[P^{-1}b \middle| P^{-1}Ab \middle| P^{-1}A^2 b \middle| \ldots \middle| P^{-1}A^{n-1}b \middle| \right]$$

$$= p^{-1}\left[b \middle| Ab \middle| A^2 b \middle| \ldots \middle| A^{n-1}\, b \right]$$

$$Q_1 \ = p^{-1}Q$$

where,

$$Q = \left[b \middle| Ab \middle| \ldots \middle| A^{n-1}\, b \right].$$

Since Q_1 and p^{-1} are both nonsingular, then Q also is non singular

As Q is non singular, its n columns are linearly independent. So that the rank of the matrix Q written as

$$r(Q) = n \qquad 3.14$$

Application 3.4. Consider the given the linear continuous-time system

$$\dot{x} = \begin{bmatrix} 0 & 1 & -2 \\ 3 & -4 & 5 \\ -6 & 7 & 8 \end{bmatrix} x + \begin{bmatrix} 0 & -1 \\ 2 & -3 \\ 4 & -5 \end{bmatrix} u$$

This is a third-order system and its controllability matrix is given by

$$Q = \left[B \middle| AB \middle| A^2 B \right]$$

$$\left[\begin{array}{cc|cc|c} 0 & -1 & -6 & 7 & \\ 2 & -3 & 12 & -10 & A^2 B \\ 4 & -5 & 46 & -55 & \end{array}\right]$$

Since the first columns are linearly independent we can conclude that the $rank\ Q = 3$. Hence there is no need to compute A^2B since it is well known from linear algebra that the row rank of a given matrix is equal to its column rank. Thus, $rank\ Q = 3 = n$ implies that the system under consideration is controllable.

Further application to illustrate when a system is uncontrollable

Application 3.5. Using the Kalman's controllability criterion, verify that the system defined by the state equations

$$\dot{x} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} x + \begin{pmatrix} 4 \\ 6 \end{pmatrix} u$$

is uncontrollable.

Solution:

Since,

$$n = 2$$

then,

$$Q = [b | Ab] = \begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix}.$$

Since,

$|Q| = 0 \Rightarrow Q$ is singular, then

$$r(Q) = 1 < 2$$

Hence the system is uncontrollable.

4. CONCLUSION

The results of this work go a long way to add a new vent in the study of controllability. The control of the semilinear system using Banach contraction mapping principle and the

Picard's iteration method showed that the system has a unique solution and because the convergence guarantees stability, hence we conclude that the solution is stable. This add new impetus in the control of semilinear systems, as the Banach contraction mapping principle is used to forge a new and far reaching contribution to the subject.

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